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Excitation of a Layered Sphere by Multiple Point-Generated Primary Fields

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Outline

- Excitation by multiple sources: background and motivation
- Mathematical formulation
- Excitation operators and overall fields
- Derivation of the exact solution of the direct problem
- Low-frequency approximations
- Numerical results
- Conclusions

Background

- T-Matrix method: A method for the solution of the direct scattering problem, that reduces the problem of identifying the exact form of the scattered fields into solving 2×2 linear systems.
- It was first introduced by *Peter Waterman* in 1969 and since, it has been established as an effective approach for scattering problems involving stratified media or periodic structures.
- Excitation by multiple sources: In many real-world problems, the overall radiation is a result of excitation by many different sources, e.g. electromagnetic activity of the brain, isotropic radiators, wind profiling SODARs, etc.
- Inverse schemes exploiting *a priori* assumptions for the nature of these sources are often used, e.g. beamforming techniques using microphone arrays (aero-acoustics) or dipole arrays (electromagnetics).

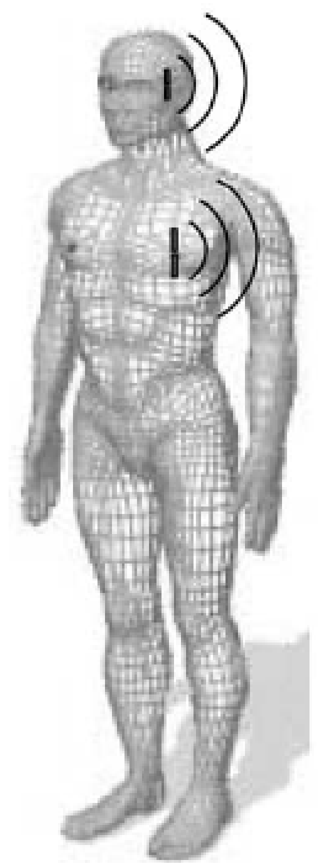
Objectives

- A layered medium excited by multiple point-generated fields: useful and realistic model for a variety of applications

- Present research objectives:
  - Formulation of the problem based on the T-Matrix approach
  - Derivation for the exact solution of the direct problem
  - Low-frequency approximations for the overall far-field and the overall scattering cross section
  - Preliminary numerical implementations

Motivating applications[1]

- Activity of the human body (e.g. brain, heart)
  - "*the field generated by the heart may be regarded as not significantly different from that of a dipole at the center of a homogeneous spherical conductor*"
    - Wilson, Bayley, *Circulation*, 1950
- Biomedical (biotelemetry, cancer treatment, hyperthermia)
  - dipole implantations in the head
    - Kiourti et al, *IEEE Trans. Biomed. Eng.*, 2012
    - Kim, Rahmat-Samii, *IEEE Trans. Microw. Theory Techn.*, 2004
- Electroencephalography and medical imaging
  - source inside the brain determined by scalp measurements
    - Sten, Lindell, *Microw. Opt. Tech. Let.*, 1992
    - Dassios, Fokas, *Inverse Problems*, 2009
  - brain imaging:
    modeling by point-dipoles inside spherical or ellipsoidal media
    - Dassios, *Lect. Notes Math.*, 2009
    - Ammari, *Introduction to Mathematics of Emerging Biomedical Imaging*, 2008



Motivating applications[2]

- Method of point sources and partial waves
  - Approximation by multiple point sources
    - Potthast, *Point Sources and Multipoles in Inverse Scattering Theory,* 2001
    - Hollmann, Wang, *Applied Optics,* 2007
- Antenna design
  - microstrip antennas, dipole arrays, RFID antennas
    - Yu, Li, Tentzeris, *Small Antennas: Miniaturization Techniques and Applications*, 2016
- Meteorology
  - Wind profiling via SODAR
    - Bradley, *Atmospheric Acoustic Remote Sensing: Principles and Applications,* 2007
    - Anderson, Ludkin, Renfrew, *J. Atmospheric Oceanic Techn.,* 2005

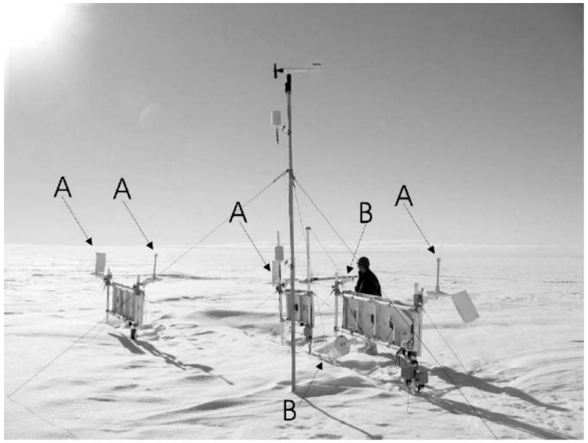
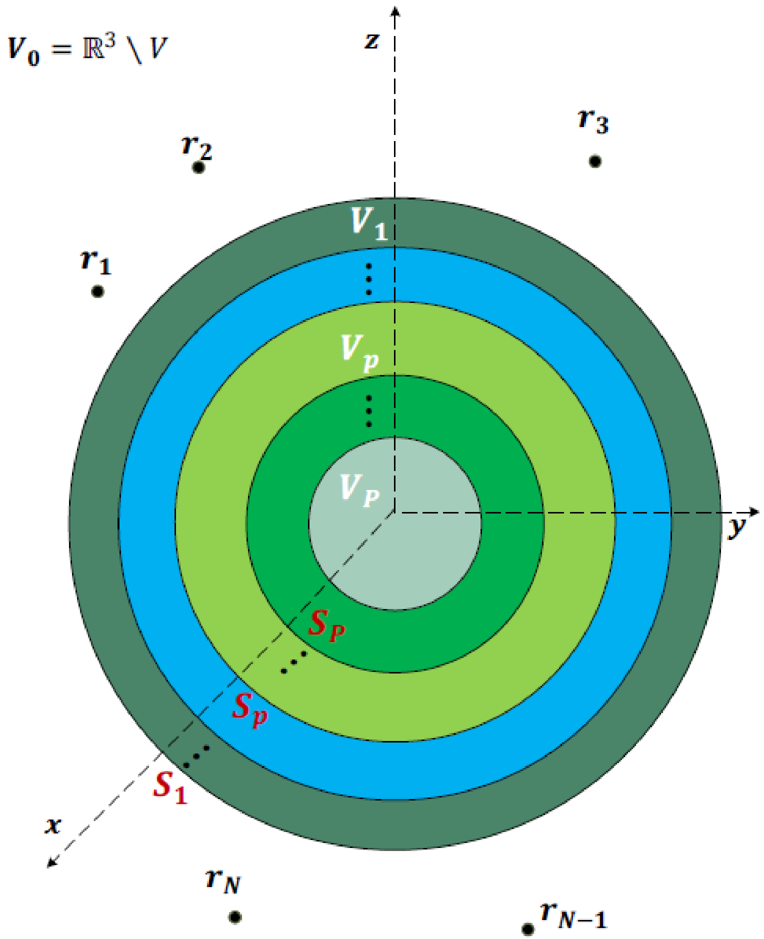


FIG. 3. Photograph of the Doppler sodar wind profiling system after 1 yr of remote opera-

Mathematical formulation[1]



N point sources located at the sphere's exterior, generate spherical acoustic fields

$$u^{\mathrm{pr}}(\mathbf{r};\mathbf{r}_j) = A_j \frac{\exp(ik_0|\mathbf{r}-\mathbf{r}_j|)}{|\mathbf{r}-\mathbf{r}_j|}, \; \mathbf{r} \neq \mathbf{r}_j$$

The layers of the sphere, are composed of materials with wavenumbers k_p and mass densities ρ_p (p=1,…,P–1).

Core V_P can be soft, hard or penetrable

Mathematical formulation[2]

- The individual and overall time-harmonic fields satisfy the scalar Helmholtz equations in the layers V_p

$$\boxed{\nabla^2 u^p(\mathbf{r};\cdot) + k_p^2 u^p(\mathbf{r};\cdot) = 0,}$$

- the transmission boundary conditions on S_p (p=1,…,P–1)

$$\boxed{\begin{gathered} u^{p-1}(\mathbf{r};\cdot) = u^p(\mathbf{r};\cdot), \quad r = a_p \\ \frac{1}{\rho_{p-1}} \frac{\partial u^{p-1}(\mathbf{r};\cdot)}{\partial n} = \frac{1}{\rho_p} \frac{\partial u^p(\mathbf{r};\cdot)}{\partial n}, \quad r = a_p \end{gathered}}$$

- and the Dirichlet or the Neumann boundary conditions on the core

$$\boxed{u^{P-1}(\mathbf{r};\cdot) = 0, \quad r = a_P} \qquad \boxed{\frac{\partial u^{P-1}(\mathbf{r};\cdot)}{\partial n} = 0, \quad r = a_P}$$

  or the transmission boundary conditions for a penetrable core

- The external field satisfies also the Sommerfeld radiation condition.

Individual Fields

- The term *individual fields* is referred to the fields induced due to a single point source exciting the scatterer. Utilizing the free-space Green's function, the individual *primary* and *secondary* fields are given, respectively, by:

$$\boxed{u_0^{\mathrm{pr}}(\mathbf{r};\mathbf{r}_j) = 4\pi \mathrm{i} k_0 A_j \begin{cases} \sum_{n=0}^{\infty}\sum_{m=-n}^{n}(-1)^m Y_n^{-m}(\hat{\mathbf{r}}_j) Y_n^m(\hat{\mathbf{r}}) \\ h_n(k_0 r) j_n(k_0 r_j),\ r > r_j \\ \sum_{n=0}^{\infty}\sum_{m=-n}^{n}(-1)^m Y_n^{m}(\hat{\mathbf{r}}_j) Y_n^{-m}(\hat{\mathbf{r}}) \\ j_n(k_0 r) h_n(k_0 r_j),\ r < r_j, \end{cases}}$$

$$\boxed{\begin{aligned} u^p(\mathbf{r};\mathbf{r}_j) = 4\pi \mathrm{i} k_0 A_j \sum_{n=0}^{\infty}\sum_{m=-n}^{n}(-1)^m Y_n^{-m}(\hat{\mathbf{r}}_j) Y_n^m(\hat{\mathbf{r}}) \\ h_n(k_0 r_j)(a_{j,n}^p j_n(k_p r) + b_{j,n}^p h_n(k_p r)), \end{aligned}}$$

Excitation Operators

- The *overall field* of layer V_p is the superposition of all individual fields induced in V_p by the external point sources. In the exterior V_0 of the sphere the external field, is the superposition of all primary and secondary fields.
- We define the following *excitation operators* which simplify the solution of the direct problem, by means of the T-Matrix approach:

$$\boxed{\begin{aligned} \mathscr{J}_{n,m}(\mathbf{x}) &= \sum_{j=1}^{N} A_j Y_n^{-m}(\hat{\mathbf{r}}_j) j_n(k_0 r_j) \mathrm{x}_j, \\ \mathscr{H}^1_{n,m}(\mathbf{x}) &= \sum_{j=1}^{N} A_j Y_n^{m}(\hat{\mathbf{r}}_j) h_n(k_0 r_j) \mathrm{x}_j, \\ \mathscr{H}^2_{n,m}(\mathbf{x}) &= \sum_{j=1}^{N} A_j Y_n^{-m}(\hat{\mathbf{r}}_j) h_n(k_0 r_j) \mathrm{x}_j, \end{aligned}}$$

$$\boxed{\mathrm{x} = (x_1, \ldots, x_N)}$$

Overall Fields and Field Expansions

With the aid of the excitation operators, the overall primary field in V_0 is given by the following formula:

$$u^{\mathrm{pr}}(\mathbf{r};\mathbf{r}_1,\ldots,\mathbf{r}_N)=4\pi i k_0\begin{cases}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}(-1)^m Y_n^m(\hat{\mathbf{r}}) \\ h_n(k_0 r)\mathscr{J}_{n,m}(\mathbf{q}), \quad r>\max[r_j] \\ \quad\vdots \\ \sum_{n=0}^{\infty}\sum_{m=-n}^{n}(-1)^m Y_n^{-m}(\hat{\mathbf{r}}) \\ j_n(k_0 r)\mathscr{H}_{n,m}^1(\mathbf{q}), \quad r<\min[r_j],\end{cases}$$

$$\mathbf{q}=(1,1,\ldots,1)$$

The secondary field in layer V_p is given by

$$u^p(\mathbf{r};\mathbf{r}_1,\ldots,\mathbf{r}_N)=4\pi i k_0\sum_{n=0}^{\infty}\sum_{m=-n}^{n}(-1)^m Y_n^m(\hat{\mathbf{r}}) \left(\mathscr{A}_{n,m}^p j_n(k_p r)+\mathscr{B}_{n,m}^p h_n(k_p r)\right).$$

$$\mathbf{a}_n^p=(a_{1,n}^p,\ldots,a_{N,n}^p)$$
$$\mathbf{b}_n^p=(b_{1,n}^p,\ldots,b_{N,n}^p)$$

$$\mathscr{A}_{n,m}^p=\mathscr{H}_{n,m}^2(\mathbf{a}_n^p) \text{ and } \mathscr{B}_{n,m}^p=\mathscr{H}_{n,m}^2(\mathbf{b}_n^p)$$

Exact Solution of the Direct Problem[1]

- The transition matrix from layer V_{p-1} to layer V_p is given by

$$\mathbf{T}_n^p = -ik_p^2 a_p^2 \begin{bmatrix} h_n'(x_p)j_n(y_p) - w_j h_n(x_p)j_n'(y_p) & h_n'(x_p)h_n(y_p) - w_j h_n(x_p)h_n'(y_p) \\ w_j j_n(x_p)j_n'(y_p) - j_n'(x_p)j_n(y_p) & w_j j_n(x_p)h_n'(y_p) - j_n'(x_p)h_n(y_p) \end{bmatrix}$$

$$x_p = k_p a_p,\ y_p = k_{p-1}a_p \text{ and } w_p = (k_{p-1}\rho_p)/(k_p\rho_{p-1})$$

- Notation for the transition matrix from layer V_p to layer V_s and boundary transition vector

$$\mathbf{\Psi}_n(x) = \left(\mathbf{T}_n^{(0\to P-1)}\right)^{\mathrm{T}} \cdot \begin{bmatrix} f_n(x) \\ g_n(x) \end{bmatrix}$$

$$\mathbf{T}_n^{(0\to p)} = \mathbf{T}_n^p \cdot \mathbf{T}_n^{p-1} \cdot \ldots \cdot \mathbf{T}_n^1$$

$$f_n(x) = \begin{cases} j_n(x), & \text{soft core} \\ j_n'(x), & \text{hard core} \end{cases}$$

$$g_n(x) = \begin{cases} h_n(x), & \text{soft core} \\ h_n'(x), & \text{hard core} \end{cases}$$

Exact Solution of the Direct Problem[2]

A straightforward implementation of the conditions on the boundaries of layers V_1 … V_{P-1} yields:

$$\boxed{\begin{bmatrix} \mathscr{A}^{P-1}_{n,m} \\ \mathscr{B}^{P-1}_{n,m} \end{bmatrix} = \mathbf{T}_n^{(0\to P-1)} \begin{bmatrix} \mathscr{H}^1_{n,m}(\mathbf{q}) \\ \mathscr{B}^0_{n,m} \end{bmatrix}}$$

For a soft or hard core the coefficient of the overall secondary field is given by:

$$\boxed{\mathscr{B}^0_{n,m} = -\frac{\Psi^1_n(k_{P-1}a_P)\,\mathscr{H}^1_{n,m}(\mathbf{q})}{\Psi^2_n(k_{P-1}a_P)}}$$

For a penetrable core we obtain:

$$\boxed{\mathscr{B}^0_{n,m} = -\frac{T^{(0\to P)}_{21,n}\;\mathscr{H}^1_{n,m}(\mathbf{q})}{T^{(0\to P)}_{22,n}}}$$

Exact Solution of the Direct Problem[3]

The coefficients of the individual external secondary fields, for a soft/hard or penetrable core, are given, respectively, by the formulas:

$$\boxed{b_{j,n}^{0} = -\frac{\Psi_n^1(k_{P-1}a_P)\,\mathscr{H}_{n,m}^1(\mathbf{h}_j)}{\Psi_n^2(k_{P-1}a_P)}}$$

$$\boxed{b_{j,n}^{0} = -\frac{T_{21,n}^{(0\to P)}\,\mathscr{H}_{n,m}^1(\mathbf{h}_j)}{T_{22,n}^{(0\to P)}}}$$

where

$$\boxed{\mathbf{h}_j = \frac{\mathbf{e}_j}{A_j Y_n^{-m}(\hat{\mathbf{r}}_j) h_n(k_0 r_j)}}$$

The overall far-field and the overall scattering cross section, are given by

$$\boxed{g(\hat{\mathbf{r}}) = 4\pi\mathrm{i}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}(-1)^m(-\mathrm{i})^n Y_n^m(\hat{\mathbf{r}})\mathscr{B}_{n,m}^0}$$

$$\boxed{\sigma = \frac{4\pi}{k_0^2}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}(2n+1)\frac{(n-m)!}{(n+m)!}\left|\mathscr{B}_{n,m}^0\right|^2}$$

Low-Frequency Approximations

Under the low-frequency assumption $k_0 a_1 \rightarrow 0$ the overall far-field and the overall scattering cross section take the forms:

$$g(\hat{\mathrm{r}}) = \kappa N S_1^0 + \kappa^2 \left(N\rho\eta (S_1^0)^2 + S_2^0 \sum_{j=1}^{N} \tau_j f_j(\theta,\phi) \right) + \kappa^3 \left(N\beta(\rho,\xi,\eta)(S_1^0)^3 - S_2^0 \sum_{j=1}^{N} f_j(\theta,\phi) + S_3^0 \sum_{j=1}^{N} \tau_j^2 F_j(\theta,\phi) \right)$$

$$F_j(\theta,\phi) = \sin 2\theta \sin 2\theta_j \cos(\phi_j - \phi) + \sin^2\theta \sin^2\theta_j \cos(2(\phi_j - \phi)) + (\cos^2\theta_j - \frac{1}{3})(3\cos^2\theta - 1))$$

$$\sigma = 4\pi a_1^2 \left[N^2 (S_1^0)^2 + (k_0 a_1)^2 \left(N^2 (S_1^0)^4 \delta(\rho,\eta,\xi) + \frac{(S_2^0)^2}{3} \left(\sum_{j=1}^{N} \tau_j^2 + 2 \sum_{j=1}^{N-1} \sum_{\nu=j+1}^{N} \tau_j \tau_\nu f_j(\theta_\nu,\phi_\nu) \right) \right) \right]$$

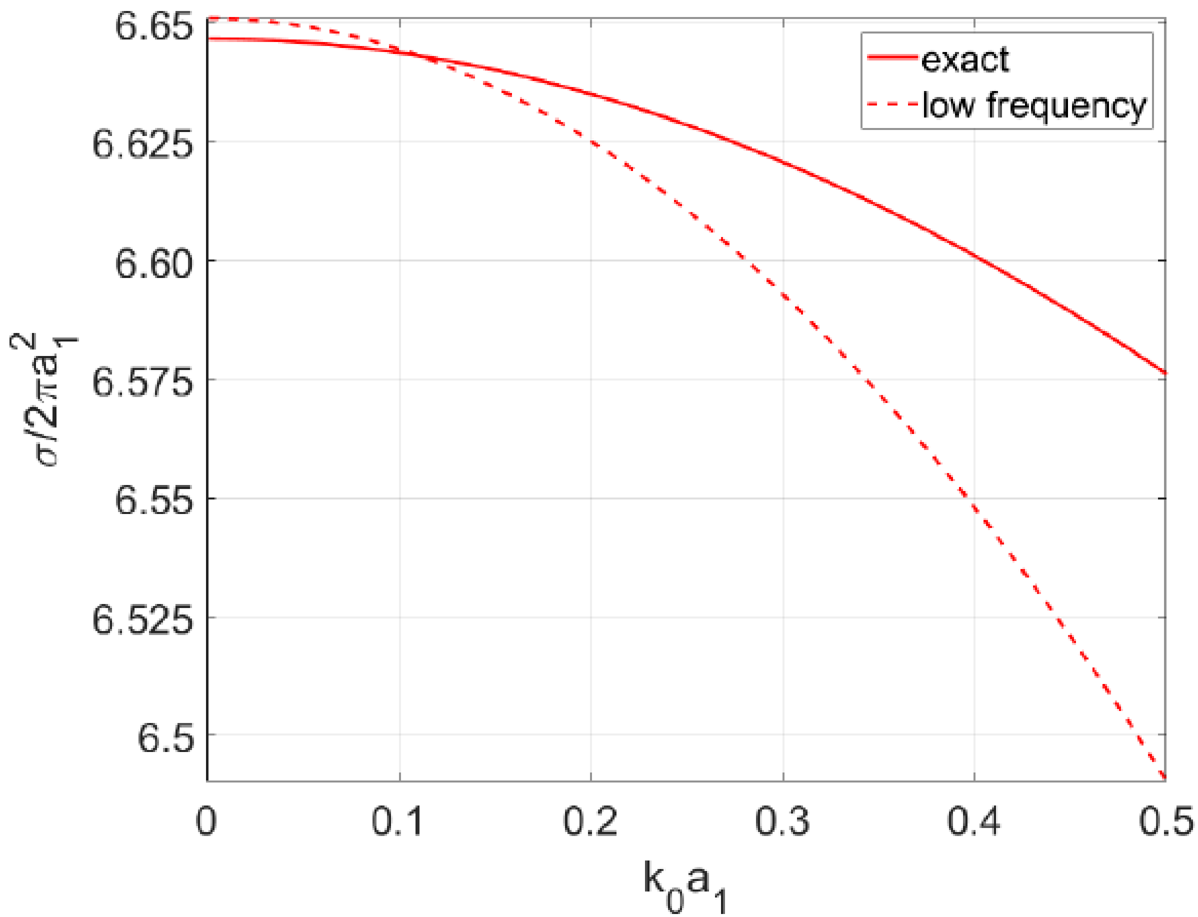
$$f_j(\theta,\phi) = \cos\theta_j \cos\theta + \sin\theta_j \sin\theta \cos(\phi_j - \phi)$$

$$S_1^0 = \frac{1}{\rho - 1 - \rho\xi}, \quad S_2^0 = \frac{\xi^3(1-\rho) + 2 + \rho}{\xi^3(1+2\rho) + 2 + 2\rho}, \quad S_3^0 = -\frac{2\xi^5(1-\rho) + 3 + 2\rho}{2\xi^5(2+3\rho) + 3 - 3\rho},$$

$$\beta(\rho,\eta,\xi) = \frac{(\rho\eta)^2(2\xi+1) - \rho\eta^2}{3\xi}, \quad \delta(\rho,\eta,\xi) = (\rho\eta)^2 \left(1 - \frac{\rho(2\xi+1) - 1}{3\xi\rho} \right)$$

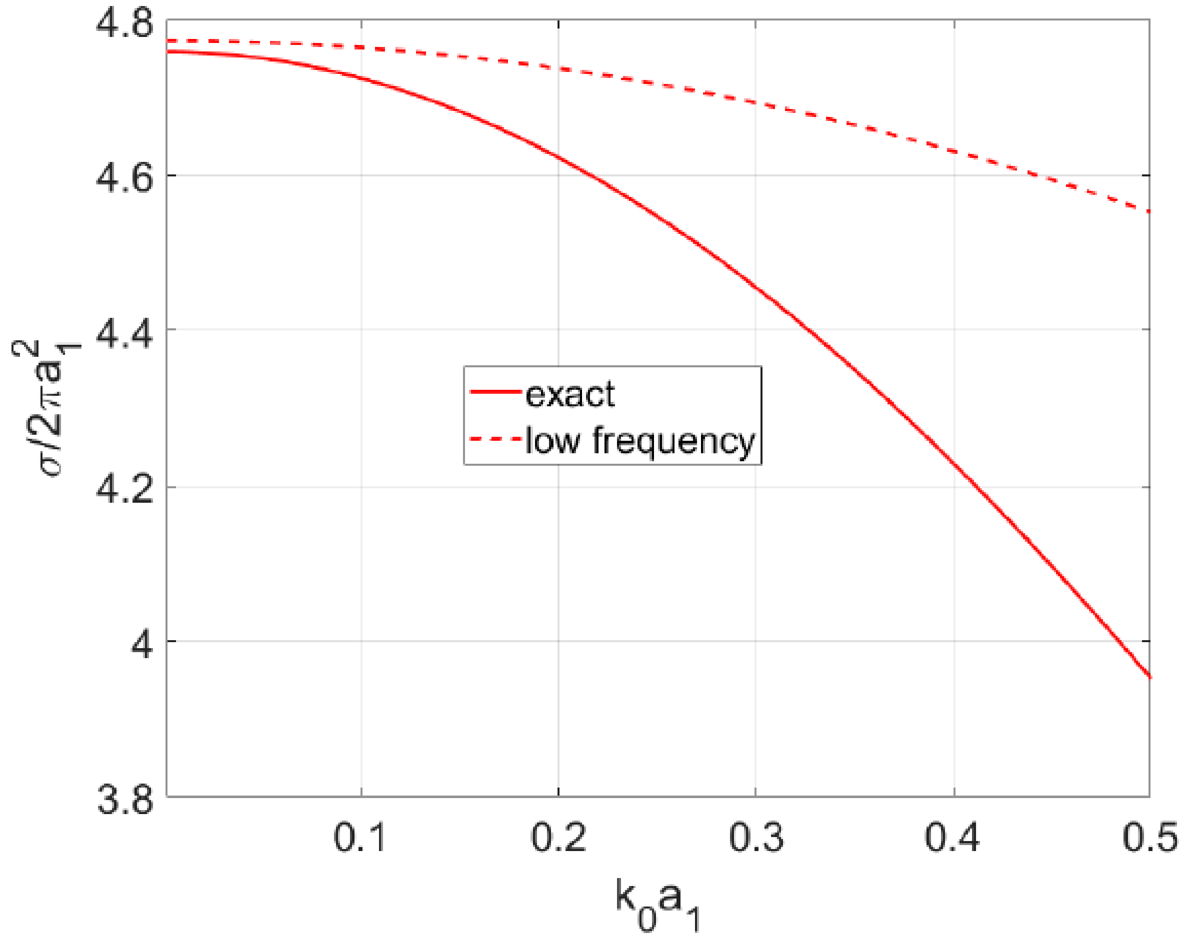
Numerical results[1]

Low-frequency approximation vs. exact scattering cross section for a 2-layered sphere with a soft core with relative mass density index *ρ=1.5* and refractive index *η=1.75*



Numerical results[2]

Low-frequency approximation vs. exact scattering cross section for a 2-layered sphere with a soft core with relative mass density index *ρ=2.5* and refractive index *η=2.25*



Conclusions

- Acoustic excitation of a layered sphere by *N* external point sources
- Motivating applications
- Mathematical formulation of the direct scattering problem and suitable definitions of fields, far-field patterns and scattering cross sections
- Overall fields, excitation operators and T-Matrix formulation
- Exact solution of the direct scattering problem
- Low-frequency approximations
- Numerical results