MATHEMATICAL THEORY OF SURFACE WAVES IN AN INHOMOGENEOUS WAVEGUIDE

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An approach based on the reduction to eigenvalue problems for operator pencils considered in Sobolev spaces was proposed by Smirnov in [1]. General theory of polynomial operator-functions called operator pencils is sufficiently well elaborated. Operator pencils were applied to the analysis of electromagnetic problems in [2,3].

Open waveguide structures were investigated by a number of authors [4,5]. However, for open (unshielded) structures, a complete theory of wave propagation is not constructed. In this case the problem becomes much more complicated (due to the non-compactness of the corresponding operators). The article deals with open inhomogeneous metal-dielectric waveguide structures i.e. the case of an unbounded exterior domain is considered. The first results on the investigation of such problems were recently obtained in [6,7] for the polarized waves propagating in a circular waveguide.

In this problem we have to analyze not the operator pencil, but an operator-function. Nevertheless, it is possible to study the properties of the operator-function in sufficient detail and obtain results on its spectrum. The discreteness of the spectrum of the problem of surface waves is proved in the article. Note that we consider waves that decrease at a distance from the waveguide (we impose the corresponding conditions at infinity). Other types of waves are not considered. This approach was used to study the shielded waveguide structures as well.

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[1] Yu.G. Smirnov, "Application of the operator pencil method in the eigenvalue problem for partially," *Doklady AN SSSR*, **312**, 1990, p. 597–599.

[2] A.L. Delitsyn, "An approach to the completeness of normal waves in a waveguide with magnitodielectric filling," *Differential Equations*, **36**, 2000, p. 695–700.

[3] A.S. Zilbergleit, Yu.I. Kopilevich, "Spectral theory of guided waves," *London: Inst. of Phys. Publ*, 1966.

[4] A.L. Delitsyn, "An approach to the completeness of normal waves in a waveguide with magnitodielectric filling," *Differential Equations*, **36**, 2000, p. 695–700.

[5] L. Levin, "Theory of waveguides," *London: Newnes-Butterworths*, 1975.

[6] Yu.G. Smirnov, E. Smolkin, "Discreteness of the spectrum in the problem on normal waves in an open inhomogeneous waveguide," *Differential Equations*, **53(10)**, 2017, p. 1168–1179.

[7] Yu.G. Smirnov, E. Smolkin, M.O. Snegur, "Analysis of the Spectrum of Azimuthally Symmetric Waves of an Open Inhomogeneous Anisotropic Waveguide with Longitudinal Magnetization," *Computational Mathematics and Mathematical Physics*, **58(11)**, 2018, pp. 1887–1901.

Consider the three-dimensional space $\mathbb{R}^3$ with the cylindrical coordinate system $O\rho\varphi z$. The space is filled with an isotropic source-free medium with permittivity $\varepsilon = \varepsilon_0 \equiv const$ and permeability $\mu = \mu_0 \equiv const$, where ε_0 and μ_0 are permittivity and permeability of vacuum. An inhomogeneous metal-dielectric waveguide with a cross-section

$$\Sigma := \{(\rho, \varphi, z) : r_0 \leqslant \rho \leqslant r, 0 \leqslant \varphi < 2\pi\}$$

and a generating line parallel to the axis Oz is placed in $\mathbb{R}^3$.
The cross section of the waveguide, which is perpendicular to its axis, consists of two concentric circles of radii r_0 and r (see Fig. .1): r is the radii of the internal (perfectly conducting) cylinder, and $r - r_0$ is the thickness of the external (dielectric) cylindrical shell. The geometry of the problem is shown in Fig. .1.

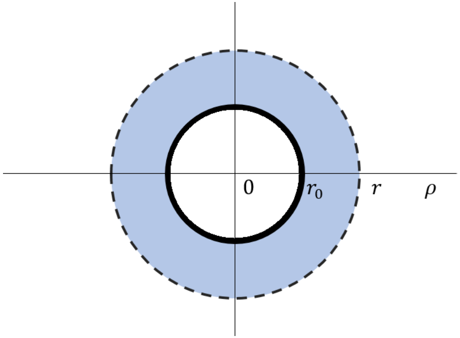

Рис. .1: Geometry of the problem.

The problem on surface waves in an inhomogeneous metal-dielectric waveguide structure is the problem of finding nontrivial running wave solutions of the homogeneous system of Maxwell equations, i.e., solutions with dependence of the from $e^{im\varphi+i\gamma z}$ on the coordinates φ and z, along which the structure is regular,

$$\begin{cases} \operatorname{rot} \mathbf{H} = -i\tilde{\varepsilon}\mathbf{E}, \\ \operatorname{rot} \mathbf{E} = i\tilde{\mu}\mathbf{H}, \end{cases} \tag{1}$$

$$\begin{gathered} \mathbf{E} = (E_\rho(\rho)\,\mathbf{e}_\rho + E_\varphi(\rho)\,\mathbf{e}_\varphi + E_z(\rho)\,\mathbf{e_z})\,e^{im\varphi+i\gamma z}, \\ \mathbf{H} = (H_\rho(\rho)\,\mathbf{e}_\rho + H_\varphi(\rho)\,\mathbf{e}_\varphi + H_z(\rho)\,\mathbf{e_z})\,e^{im\varphi+i\gamma z}, \end{gathered} \tag{2}$$

with boundary conditions for tangential electric components on perfectly conducting surfaces ($\rho = r_0$)

$$E_\varphi(r_0) = 0,\ E_z(r_0) = 0, \tag{3}$$

transmission conditions for tangential electric and magnetic components on surfaces of "jump" of permittivity and permeability ($\rho = r$)

$$[E_\varphi]|_r = 0,\, [E_z]|_r = 0,\, [H_\varphi]|_r = 0,\, [H_z]|_r = 0, \tag{4}$$

the finite energy condition

$$\int\limits_{r_0}^{\infty} (\widetilde{\varepsilon}|\mathbf{E}|^2 + \widetilde{\mu}|\mathbf{H}|^2)d\rho < \infty, \tag{5}$$

and the radiation condition at infinity: the electromagnetic field decays as $o(\rho^{-1/2})$ for $\rho \to \infty$.

The Maxwell system (1) is written in the normalized form. The passage to dimensionless variables has been carried out; namely, $k_0\rho \to \rho,\ \gamma \to \frac{\gamma}{k_0},\ \sqrt{\dfrac{\mu_0}{\varepsilon_0}}\mathbf{H} \to \mathbf{H},\ \mathbf{E} \to \mathbf{E},$ where $k_0^2 = \omega\varepsilon_0\mu_0$. (The time factor $e^{-i\omega t}$ is omitted everywhere.)
We assume that the permittivity and permeability in the entire space have the form

$$\widetilde{\varepsilon} = \begin{cases} \varepsilon(\rho), & r_0 \le \rho \le r, \\ 1, & \rho > r, \end{cases} \quad \text{and } \widetilde{\mu} = \begin{cases} \mu(\rho), & r_0 \le \rho \le r. \\ 1, & \rho > r. \end{cases}$$

We also assume that $\varepsilon(\rho) > 1$ and $\mu(\rho) > 1$ are twice continuously differentiable function on the segment $[r_0, r]$, i.e., $\varepsilon(\rho) \in C^2[r_0, r]$ and $\mu(\rho) \in C^2[r_0, r]$, $\operatorname{Im}\varepsilon(\rho) = 0,\ \operatorname{Im}\mu(\rho) = 0.$

The problem on normal waves is an eigenvalue problem for the Maxwell equations with spectral parameter γ, which is the normalized propagation constant of GL.
Rewrite system (1) in the expanded form and express the functions $E_\rho, H_\rho, E_\varphi, H_\varphi$ via the functions E_z and H_z from the first, second, fourth, and fifth equations in system (1)

$$\begin{aligned} E_\rho &= \frac{m\widetilde{\mu}H_z - i\gamma\rho E_z'}{\rho\widetilde{\kappa}^2}, \qquad & H_\rho &= -\frac{i\gamma\rho H_z' + m\widetilde{\varepsilon}E_z}{\rho\widetilde{\kappa}^2}, \\ E_\varphi &= \frac{\gamma m E_z + i\rho\widetilde{\mu}H_z'}{\rho\widetilde{\kappa}^2}, \qquad & H_\varphi &= \frac{\gamma m H_z - i\rho\widetilde{\varepsilon}E_z'}{\rho\widetilde{\kappa}^2}, \end{aligned} \tag{6}$$

where $\widetilde{\kappa}^2 = \gamma^2 - \widetilde{\varepsilon}\widetilde{\mu}$.
It follows from Eqs. (6) that the normal wave field in the waveguide can be represented with the use of two scalar functions

$$u_e := iE_z(\rho),\ u_m := H_z(\rho). \tag{7}$$

Thus, the problem has been reduced to finding the longitudinal components u_e and u_m of the electric and magnetic fields. Throughout the following, $(\,\cdot\,)'$ stands for differentiation with respect to ρ.
We have the following eigenvalue problem for the longitudinal field components u_e and u_m: find $\gamma \in \mathbb{C}$ such that, for given $m \in \mathbb{Z}$, there exist nontrivial solutions of the system of differential equations

$$\begin{cases} \left(\dfrac{\widetilde{\varepsilon}\rho}{\widetilde{\kappa}^2}u_e'\right)' - \dfrac{\widetilde{\varepsilon}}{\rho}\left(\rho^2 + \dfrac{m^2}{\widetilde{\kappa}^2}\right)u_e = \gamma m \dfrac{(\widetilde{\varepsilon}\widetilde{\mu})'}{\widetilde{\kappa}^2}u_m, \\ \left(\dfrac{\widetilde{\mu}\rho}{\widetilde{\kappa}^2}u_m'\right)' - \dfrac{\widetilde{\mu}}{\rho}\left(\rho^2 + \dfrac{m^2}{\widetilde{\kappa}^2}\right)u_m = \gamma m \dfrac{(\widetilde{\varepsilon}\widetilde{\mu})'}{\widetilde{\kappa}^2}u_e, \end{cases} \tag{8}$$

satisfying the boundary conditions for $\rho = r_0$

$$u_e(r_0) = 0,\ u_m'(r_0) = 0, \tag{9}$$

transmission conditions for $\rho = r$

$$\begin{gathered} [u_e]\big|_r = 0,\ [u_m]\big|_r = 0, \\ \gamma m \left[\frac{u_e}{\widetilde{\kappa}^2}\right]\bigg|_r - \left[\frac{\rho\widetilde{\mu}u_m'}{\widetilde{\kappa}^2}\right]\bigg|_r = 0, \\ \gamma m \left[\frac{u_m}{\widetilde{\kappa}^2}\right]\bigg|_r - \left[\frac{\rho\widetilde{\varepsilon}u_e'}{\widetilde{\kappa}^2}\right]\bigg|_r = 0, \end{gathered} \tag{10}$$

and also with the condition of boundedness of the field in any finite domain and the decay condition at infinity.

Once we determine the longitudinal field components u_e and u_m by solving problem (8)–(10), we can find the transverse components by formulas (6). The field $(\mathbf{E}, \mathbf{H})$ thus obtained satisfies all conditions (1)–(5). The equivalence of reduction to problem (8)–(10) is not valid only for $\gamma^2 = \widetilde{\varepsilon}\widetilde{\mu}$; in this case it is necessary to study the system (1) directly.

For $\rho > r$, we have $\widetilde{\varepsilon} = 1,\ \widetilde{\mu} = 1$; in view of the condition at infinity, we obtain a solution of the system (8) in the form

$$\begin{cases} u_e(\rho; \gamma, m) = C_1 K_m(\kappa_1 \rho), \\ u_m(\rho; \gamma, m) = C_2 K_m(\kappa_1 \rho), \end{cases} \tag{11}$$

where $\kappa_1^2 = \gamma^2 - 1$ and K_m is the modified Bessel function (the Macdonald function) [8]; C_1 and C_2 are constants.

The function $\kappa_1(\gamma)$ is analytic in the domain

$$\mathbb{C} \backslash \Lambda_K, \text{ where } \Lambda_K := \{\gamma :\ \operatorname{Im} \gamma^2 = 0,\ \gamma^2 \leq 1\}.$$

For $r_0 \leq \rho \leq r$, we have $\widetilde{\varepsilon} = \varepsilon(\rho)$ and $\widetilde{\mu} = \mu(\rho)$, and from system (8) we obtain the system of differential equations

$$\begin{cases} \left(\dfrac{\varepsilon\rho}{\kappa^2} u_e'\right)' - \dfrac{\varepsilon}{\rho}\left(\rho^2 + \dfrac{m^2}{\kappa^2}\right) u_e = \gamma m \dfrac{(\varepsilon\widetilde{\mu})'}{\kappa^2} u_m, \\ \left(\dfrac{\mu\rho}{\kappa^2} u_m'\right)' - \dfrac{\widetilde{\mu}}{\rho}\left(\rho^2 + \dfrac{m^2}{\kappa^2}\right) u_m = \gamma m \dfrac{(\varepsilon\mu)'}{\kappa^2} u_e \end{cases} \tag{12}$$

Definition 1

If for given m there exist nontrivial functions u_e and u_m corresponding to some $\gamma \in \mathbb{C}$ such that these functions are the solutions (11) for $\rho > r$, are a solution of system (12) for $r_0 \leq \rho \leq r$, and satisfy the transmission conditions (10), then γ is called a characteristic number of problem P_m.

Definition 2

The pair u_e and u_m, $|u_e|^2 + |u_m|^2 \not\equiv 0$, will be called an eigenvector of problem P_m corresponding to the characteristic number $\gamma \in \mathbb{C}$.

We will find the solutions u_e and u_m of the problem P_m in Sobolev spaces

$$H_0^1(r_0, r) = \left\{ f : f \in H^1(r_0, r),\ f(r_0) = 0 \right\} \text{ and } H^1(r_0, r),$$

with the inner product and the norm

$$(f, g)_1 = \int_{r_0}^{r} \left(f' \overline{g}' + f \overline{g} \right) d\rho,$$

and $\|f\|_1^2 = (f, f)_1 = \int_{r_0}^{r} \left(|f'|^2 + |f|^2 \right) d\rho.$

Let us give variational formulation of the problem P_m. We multiply equations (12) by arbitrary test functions $v_e \in H_0^1(r_0, r)$ and $v_m \in H^1(r_0, r)$ (we can assume that these functions are continuously differentiable in (r_0, r)), next we apply Green's formula, taking into account boundary condition for $\rho = r_0$ and $\rho = r$, we obtain a *variational relation*

$$\gamma^4 \int\limits_{r_0}^{r} (u_e \overline{v}_e + u_m \overline{v}_m) d\rho + \gamma^2 \int\limits_{r_0}^{r} (u_e' \overline{v}_e' + u_m' \overline{v}_m') d\rho +$$

$$+ \gamma^2 \int\limits_{r_0}^{r} (p_e u_e' \overline{v}_e + p_m u_m' \overline{v}_m) d\rho + \gamma^2 \int\limits_{r_0}^{r} r_1 (u_e \overline{v}_e + u_m \overline{v}_m) d\rho -$$

$$- \int\limits_{r_0}^{r} \mu\varepsilon (u_e' \overline{v}_e' + u_m' \overline{v}_m') d\rho - \int\limits_{r_0}^{r} (\mu\varepsilon)' (u_e' \overline{v}_e + u_m' \overline{v}_m) d\rho +$$

$$+ \int\limits_{r_0}^{r} (q_e u_e' \overline{v}_e + q_m u_m' \overline{v}_m) d\rho + \int\limits_{r_0}^{r} r_2 (u_e \overline{v}_e + u_m \overline{v}_m) d\rho +$$

$$+ \frac{\kappa_r^2}{\varepsilon(r)} \left(\frac{\gamma m}{r} \frac{\chi}{\kappa_1^2} u_m(r) - \frac{\kappa_r^2}{\kappa_1} \frac{K_m'(\kappa_1 r)}{K_m(\kappa_1 r)} u_e(r) \right) \overline{v}_e(r) +$$

$$+ \frac{\kappa_r^2}{\mu(r)} \left(\frac{\gamma m}{r} \frac{\chi}{\kappa_1^2} u_e(r) - \frac{\kappa_r^2}{\kappa_1} \frac{K_m'(\kappa_1 r)}{K_m(\kappa_1 r)} u_m(r) \right) \overline{v}_m(r) -$$

$$- \gamma \int\limits_{r_0}^{r} (f_e u_m \overline{v}_e + f_m u_e \overline{v}_m) d\rho = 0, \quad (13)$$

Let $H = H_0^1(r_0,\ r) \times H^1(r_0,\ r)$ be the Cartesian product of Hilbert spaces with inner product and norm

$$(\mathbf{u}, \mathbf{v}) = (u_1,\ v_1)_1 + (u_2,\ v_2)_1,\ \|\mathbf{u}\|^2 = \|u_1\|_1^2 + \|u_2\|_1^2;$$

$$\mathbf{u} = (u_1,\ u_2)^T,\ \mathbf{v} = (v_1,\ v_2)^T.$$

Then the integrals occurring in (13) can be viewed as sesquilinear forms over the field $\mathbb{C}$ defined on the space H and depending on the arguments $\mathbf{u} = (u_e,\ u_m)^T$ and $\mathbf{v} = (\overline{v}_e,\ \overline{v}_m)^T$ These forms t define some bounded linear operators $\mathrm{T} : H \to H$ by the formula [8]

$$\mathrm{t}(\mathbf{u},\ \mathbf{v}) = (\mathrm{T}\mathbf{u},\ \mathbf{v}),\ \forall \mathbf{v} \in H,$$

provided that the forms themselves are bounded.

[8] T. Kato, "Perturbation Theory for Linear Operators," *New York: Springer-Verlag*, 1980.

Consider the following sesquilinear forms and the corresponding operators ($\forall \mathbf{v} \in H$):

$$\mathrm{k}(\mathbf{u},\ \mathbf{v}) := \int\limits_{r_0}^{r} (u_e \overline{v}_e + u_m \overline{v}_m) d\rho = (\mathrm{K}\mathbf{u}, \mathbf{v}),$$

$$\kappa_1(\mathbf{u},\ \mathbf{v}) := \int\limits_{r_0}^{r} (r_1 - 1)(u_e \overline{v}_e + u_m \overline{v}_m) d\rho = (\mathrm{K}_1\mathbf{u}, \mathbf{v}),$$

$$\mathrm{k}_2(\mathbf{u},\ \mathbf{v}) := \int\limits_{r_0}^{r} (r_2 - \mu\varepsilon)(u_e \overline{v}_e + u_m \overline{v}_m) d\rho = (\mathrm{K}_2\mathbf{u}, \mathbf{v}),$$

$$\widetilde{\mathrm{k}}(\mathbf{u},\ \mathbf{v}) := \int\limits_{r_0}^{r} (f_e u_m \overline{v}_e + f_m u_e \overline{v}_m) d\rho = (\widetilde{\mathrm{K}}\mathbf{u}, \mathbf{v}),$$

$$\mathrm{a}_1(\mathbf{u},\ \mathbf{v}) := \int\limits_{r_0}^{r} (u_e' \overline{v}_e' + u_m' \overline{v}_m' + u_e \overline{v}_e + u_m \overline{v}_m) d\rho = (\mathrm{I}\mathbf{u}, \mathbf{v}),$$

$$\mathrm{a}_2(\mathbf{u},\ \mathbf{v}) := \int\limits_{r_0}^{r} \mu\varepsilon(u_e' \overline{v}_e' + u_m' \overline{v}_m' + u_e \overline{v}_e + u_m \overline{v}_m) d\rho = (\mathrm{A}\mathbf{u}, \mathbf{v}),$$

$$\mathrm{b}_1(\mathbf{u},\ \mathbf{v}) := \int_{r_0}^{r} (p_e u_e' \overline{v}_e + p_m u_m' \overline{v}_m) d\rho = (\mathrm{B}_1\mathbf{u}, \mathbf{v}),\ \forall \mathbf{v} \in H,$$

$$\mathrm{b}_2(\mathbf{u},\ \mathbf{v}) := \int_{r_0}^{r} (\mu\varepsilon)'(u_e' \overline{v}_e + u_m' \overline{v}_m) d\rho = (\mathrm{B}_2\mathbf{u}, \mathbf{v}),$$

$$\mathrm{b}_3(\mathbf{u},\ \mathbf{v}) := \int_{r_0}^{r} (q_e u_e' \overline{v}_e + q_m u_m' \overline{v}_m) d\rho = (\mathrm{B}_3\mathbf{u}, \mathbf{v}),$$

$$\mathrm{s}(\mathbf{u},\ \mathbf{v}) = \frac{\kappa_r^2}{\varepsilon(r)} \left(\frac{\gamma m}{r} \frac{\chi}{\kappa_1^2} u_m(r) - \frac{\kappa_r^2}{\kappa_1} \frac{K_m'(\kappa_1 r)}{K_m(\kappa_1 r)} u_e(r) \right) \overline{v}_e(r) + \\ + \frac{\kappa_r^2}{\mu(r)} \left(\frac{\gamma m}{r} \frac{\chi}{\kappa_1^2} u_e(r) - \frac{\kappa_r^2}{\kappa_1} \frac{K_m'(\kappa_1 r)}{K_m(\kappa_1 r)} u_m(r) \right) \overline{v}_m(r) = (\mathrm{S}\mathbf{u}, \mathbf{v}).$$

The variational problem (13) can be written in the operator form

$$\mathrm{N}(\gamma)\mathbf{u} := (\gamma^4 \mathrm{K} + \gamma^2 \left(\mathrm{K}_1 + \mathrm{B}_1 + \mathrm{I} - \gamma \widetilde{\mathrm{K}} + \mathrm{K}_2 - \mathrm{A} - \mathrm{B}_2 + \mathrm{B}_3 + \mathrm{S}(\gamma)\mathbf{u} = 0. \quad (14)$$

We have reduced the problem on normal waves to the study of spectral properties of the operator function N. In this connection, we first consider the properties of the operators introduced in the preceding section. The validity of Lemmas and Theorems demonstrated in [9]

Lemma 3

The bounded operator $\mathrm{A} : H \to H$ *is positive definite* $\mathrm{A} \geq \gamma_*^2 \mathrm{I}$, *where* $0 < \gamma_* = \min\limits_{r_0 \leq \rho \leq r} \sqrt{\mu(\rho)\varepsilon(\rho)}$.

Lemma 4

The bounded operators $\mathrm{K},\ \mathrm{K}_1,\ \mathrm{K}_2$ *and* $\widetilde{\mathrm{K}} : H \to H$ *are compact, and* $\mathrm{K} > 0$.

Lemma 5

The operators $\mathrm{B}_1,\ \mathrm{B}_2$ *and* $\mathrm{B}_3 : H \to H$ *are compact.*

Lemma 6

The operator $\mathrm{S} : H \to H$ *is compact.*

[9] Yu.G. Smirnov, E. Smolkin, "Discreteness of the spectrum in the problem on normal waves in an open inhomogeneous waveguide," *Differential Equations*, **53(10)**, 2017, p. 1168–1179.

Lemma 7

The operator $\gamma^2\mathrm{I} - \mathrm{A} : H \to H$ *is bounded and continuously invertible in the domain*

$$\mathbb{C}\backslash\Lambda_E \text{ and } \Lambda_E := \{\gamma : \ \mathrm{Im}\,\gamma = 0, \ \gamma_* \leq |\,\mathrm{Re}\,\gamma| \leq \ \gamma^*\},$$

where $0 < \gamma^* = \max\limits_{r_0 \leq \rho \leq r} \sqrt{\mu(\rho)\varepsilon(\rho)}$.

Lemma 8

There exists a $\widetilde{\gamma} \in \mathbb{R}$ *such that the operator* $\mathrm{N}(\widetilde{\gamma})$ *is continuously invertible; i.e., the resolvent set* $\varrho(\mathrm{N}) := \{\gamma : \exists\, \mathrm{N}^{-1}(\gamma) : H \to H\}$ *of the operator function* $\mathrm{N}(\widetilde{\gamma})$ *is nonempty,* $\varrho(\mathrm{N}) \neq \varnothing$.

Theorem 9

The operator function $\mathrm{N}(\gamma) : H \to H$ *is bounded, holomorphic, and Fredholm in the domain* $\Lambda = \mathbb{C}\backslash\,(\Lambda_K \cup \Lambda_E)$

Theorem 10

The spectrum of the operator function $\mathrm{N}(\gamma) : H \to H$ *is discrete in the domain* Λ*; i.e., this function has finitely many characteristic points of finite algebraic multiplicity on any compact set* $K_0 \subset \Lambda$.

We have reduced the boundary eigenvalue problem for the Maxwell equations describing surface waves in a dielectric waveguide to an eigenvalue problem for an operator-function. We have proved fundamental properties of the spectrum of normal waves including the discreteness and a statement describing localization of eigenvalues of the operator-function on the complex plane.

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