# DIFFRACTION OF TE POLARISED ELECTROMAGNETIC WAVES BY A NONLINEAR METAMATERIAL WAVEGUIDE

MAXIM SNEGUR AND EUGENE SMOLKIN

Penza State University

The electromagnetic wave diffraction by homogeneous [1] or inhomogeneous (see e.g. [2]) cylindrical metal-dielectric bodies filled with linear medium has been studied intensively since the 1940s. The case of nonlinear filling still constitutes an unsloved problem. A progress here is associated with recently developed techniques (see [3][4]) for the analysis of nonlinear boundary value problems for the Maxwell's and Helmholtz equations.
In 1967, Russian physicist V. G. Veselago predicted an extraordinary electromagnetic (EM) wave phenomenon which is related to materials with a simultaneously negative permittivity and negative permeability [5]. He hypothetically created a lossless meta-material and showed the extraordinary properties of this material which is not found in nature, in particular, negative group velocity, negative refraction, the reversal of the Doppler effect and Cherenkov radiation.
The present study focuses on the analysis of the diffraction of TE waves by an open waveguide, a Goubau line (GL) (see ), with a nonlinear metamaterial medium. Metamaterial is an artificial material with negative permittivity and negative permeability. The nonlinearity is expressed by the Kerr law. The main task which we resolve is to elaborate mathematically correct problem statements for nonlinear differential equations that enable one to introduce and investigate the problem of diffraction. Such problems (the scattering by cylinders covered with nonlinear materials) finds some applications in cloaking devices.

[1] R. W. P. King, Tai Tsun Wu, "The Scattering and Diffraction of Waves," *Harvard University Press, London, UK*, 1959.

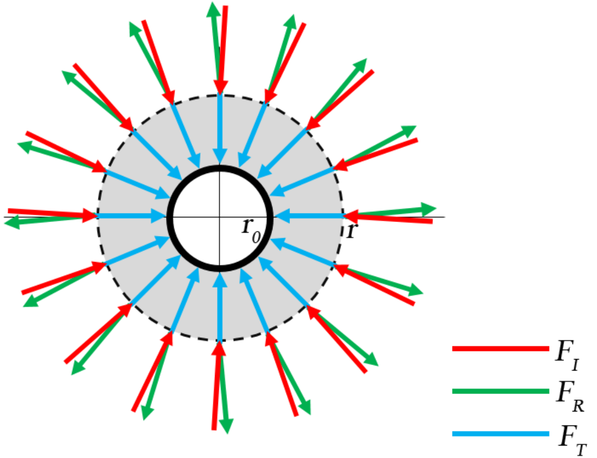
[2] Y. Miyazaki, "Scattering and diffraction of electromagnetic waves by inhomogeneous dielectric cylinder," *Radio Science*, **16**, 1981, pp. 1009–1014.

[3] D.V. Valovik, E. Smolkin, "Calculation of the propagation constants of inhomogeneous nonlinear double-layer circular cylindrical waveguide by means of the Cauchy problem method," *Journal of Communications Technology and Electronics*, **58**, 2013, pp. 759–767.

[4] Yu.G. Smirnov, E. Smolkin, "On the existence of non-polarized azimuthal-symmetric electromagnetic waves in circular dielectric waveguide filled with nonlinear isotropic homogeneous medium," *Wave Motion*, **77**, 2018, pp. 77–90.

[5] V.G. Veselago, "The electrodynamics of substances with simultaneously negative values of $\varepsilon$ and $\mu$," *Sov. Phys. Uspekhi*, **10**, 1968, pp. 509–514.

The cross section of the waveguide under study perpendicular to its axis consists of two concentric circles of radii $r_0$ and $r$ (see Fig. 1): $r_0$ is the radii of the internal (perfectly conducting) cylinder, and $r - r_0$ is the thickness of the external (dielectric) cylindrical shell.




;

Рис. .1: Waveguide $\Sigma$, where $r_0$ and $r$ are the radii of the internal (perfectly conducting) and external (dielectric) cylinders, respectively.

The complex amplitudes $\mathbf{E}$, $\mathbf{H}$ of the electromagnetic field satisfy Maxwell's equations

$$\begin{cases} \operatorname{rot}\mathbf{H} = -i\omega\varepsilon_0\widetilde{\varepsilon}\mathbf{E}, \\ \operatorname{rot}\mathbf{E} = i\omega\mu\mathbf{H}, \end{cases} \tag{1}$$

have zero tangential components of the electric field on the perfectly conducting surface $\rho = r_0$ and continuous tangential field components on the media interface $\rho = r$; here $\omega$ is the circular frequency. We assume that the permittivity in the entire space has the form $\widetilde{\varepsilon}\varepsilon_0$, where

$$\widetilde{\varepsilon} = \begin{cases} -\varepsilon^2 + \widetilde{\alpha}|\mathbf{E}|^2, & r_0 \le \rho \le r, \\ 1, & \rho > r, \end{cases} \tag{2}$$

and $|\mathbf{E}|^2 = \left|(\mathbf{E}e^{-i\omega t}, \mathbf{e}_\rho)\right|^2 + \left|(\mathbf{E}e^{-i\omega t}, \mathbf{e}_\varphi)\right|^2 + \left|(\mathbf{E}e^{-i\omega t}, \mathbf{e}_z)\right|^2$;$(\cdot,\cdot)$ is the Euclidean inner product; $\varepsilon^2, \widetilde{\alpha}$ are real positive constants.

Let us consider TE-polarized waves in the harmonic mode, according to [**?**],

$$\mathbf{E}e^{-i\omega t} = e^{-i\omega t}(0, E_\varphi, 0)^T,\ \mathbf{H}e^{-i\omega t} = e^{-i\omega t}(H_\rho, 0, H_z)^T,$$

where $\mathbf{E}, \mathbf{H}$ are complex amplitudes,

$$E_\varphi = \mathrm{E}_\varphi(\rho)e^{i\gamma z}, H_\rho = \mathrm{H}_\rho(\rho)e^{i\gamma z}, H_z = \mathrm{H}_z(\rho)e^{i\gamma z} \tag{3}$$

and $\gamma$ is a given quantity.
Let $k_0^2 := \omega^2\mu\varepsilon_0$. Substituting components (3) into (1) and using the notation $u(\rho) := \mathrm{E}_\varphi(\rho)$ we obtain

$$\left(\rho^{-1}\left(\rho u\right)'\right)' + \left(k_0^2\widetilde{\varepsilon} - \gamma^2\right)u = 0, \tag{4}$$

where $\widetilde{\varepsilon}$ is defined by formula (2).
We assume that function $u$ is sufficiently smooth,

$$u(\rho) \in C^1\left[r_0, +\infty\right) \cap C^2(r_0, r) \cap C^2(r, +\infty).$$

In the domain $r_0 \le \rho \le r$ equation (4) takes the form

$$\left(\rho u'\right)' - \left(\rho k_1^2 + \rho^{-1}\right)u = -\alpha\rho u^3, \tag{5}$$

where $\alpha := k_0^2\widetilde{\alpha}$, $k_1^2 := k_0^2\varepsilon^2 + \gamma^2$. In the domain $\rho > r$ equation (4) becomes

$$\left(\rho u'\right)' - \left(\rho k_2^2 + \rho^{-1}\right)u = 0, \tag{6}$$

where $k_2^2 := \gamma^2 - k_0^2$.

For $\rho > r$, the solution to equation (6) must be written in the following form

$$u = \widetilde{A}I_1(k_2\rho) + \widetilde{C}K_1(k_2\rho), \quad \rho > r. \tag{7}$$

The incident field is determined by

$$u_I(\rho) = \widetilde{A}I_1(k_2\rho), \tag{8}$$

where $I_1$ is the modified Bessel function (Infeld function) [6] and $F_I = \widetilde{A}I_1(k_2r)$ is the amplitude of the incident field (for $\rho = r$). The reflected field satisfies the radiation conditions of decay at infinity and therefore can be taken in the following form at $\rho > r$

$$u_R(\rho) = \widetilde{C}K_1(k_2\rho), \tag{9}$$

where $K_1$ is the modified Bessel function (Macdonald function) and constant $F_R = \widetilde{C}K_1(k_2r)$ is the amplitude of the reflected field (for $\rho = r$). The total field in the region $\rho > r$ is a superposition of the incident, $u_I$, and reflected, $u_R$, fields,

$$u = u_I + u_R, \ \rho > r. \tag{10}$$

[6] M. Abramowitz, I.A. Stegun, "Handbook of Mathematical Functions," *National Bureau of Standards, Washington, USA*, 1972.

The amplitude of transmitted field $F_T$ (for $\rho = r$) is a sum of amplitudes of the incident, $F_I$, and reflected, $F_R$, fields,

$$F_T = F_I + F_R.$$

Transmission conditions for the functions $u$ and $u'$ result from the continuity conditions for the tangential field components ($E_\varphi$ and $H_z$) and have the form

$$\begin{aligned} [u]\big|_{\rho=r} = \left[u'\right]\big|_{\rho=r} &= 0, \\ u\big|_{\rho=r_0} &= 0, \end{aligned} \tag{11}$$

where $[v]\big|_{\rho=s} = \lim\limits_{\rho\to s-0} v\,(\rho) - \lim\limits_{\rho\to s+0} v\,(\rho)$ is the jump in the limit values of the function at a point $s$.

Formulate the diffraction problem (problem $P$): *to find amplitude $F_R$ of the reflected field such that, for the given amplitude $F_I$ of the incident field, there are nonzero function $u(\rho)$ defined by formula* (10) *for $\rho > r$ that solve the ordinary differential equation* (5) *for $r_0 < \rho < r$ and satisfy transmission conditions* (11).

For the numerical solution of Problem $P$ a method based on the solution to the auxiliary Cauchy problem is proposed which makes it possible in particular to determine and plot the amplitude of the reflected field, $F_R$, with respect to the amplitude of the incident field, $F_I$.
Consider the Cauchy problem for the equation (6) with the following initial conditions

$$
\begin{aligned}
u\,(r) &= u_I(r) + u_R(r) = F_I + F_R,\\
u'\,(r) &= u_I'(r) + u_R'(r) = F_I \frac{I_1'(k_2 r)}{I_1(k_2 r)} + F_R \frac{K_1'(k_2 r)}{K_1(k_2 r)}.
\end{aligned}
\tag{12}
$$

To justify the solution technique, we use classical results of the theory of ordinary differential equations concerning the existence and uniqueness of the solution to the Cauchy problem and continuous dependence of the solution on parameters.
Using the transmission condition on the boundary $\rho = r_0$, we obtain the following dispersion equation

$$
\Delta\,(F_I, F_R) \equiv u\,(r_0) = 0, \tag{13}
$$

where $\Delta\,(F_I, F_R)$ is determined explicitly and quantity $u\,(r_0)$ is obtained from the solution to the Cauchy problem for fixed values of $F_I$ and $F_R$.

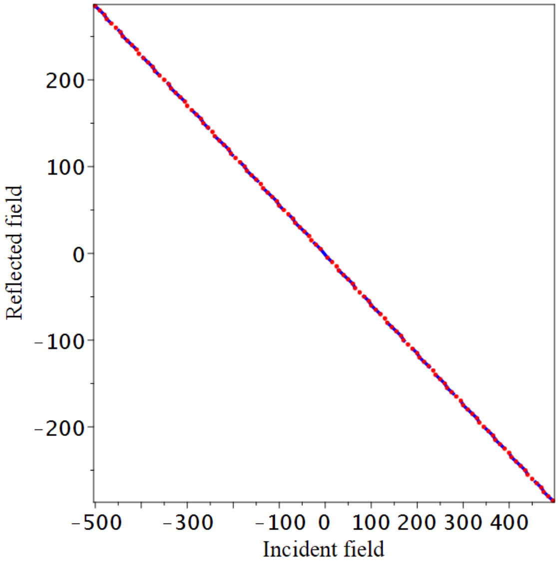



Рис. .2: Amplitude of the reflected field $F_R$ vs amplitude of the incident field $F_I$ in the linear ($\alpha = 0$, blue) and nonlinear (red) cases. The values of parameters are $\gamma = 1.15,\ k_0^2 = 1,\ r_0 = 2,\ r = 4,\ \varepsilon^2 = 4,\ \alpha = 10^{-5}$.

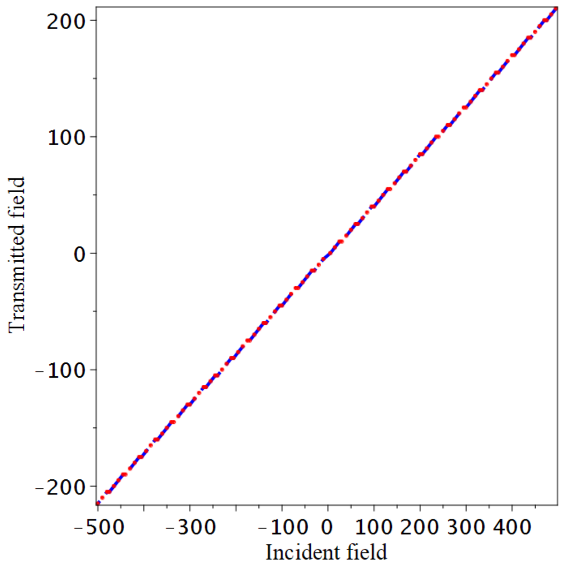



Рис. .3: Amplitude of the transmitted field $F_T$ vs amplitude of the incident field $F_I$ in the linear ($\alpha = 0$, blue) and nonlinear (red) cases. The values of parameters are $\gamma = 1.15,\ k_0^2 = 1,\ r_0 = 2,\ r = 4,\ \varepsilon^2 = 4,\ \alpha = 10^{-5}$.

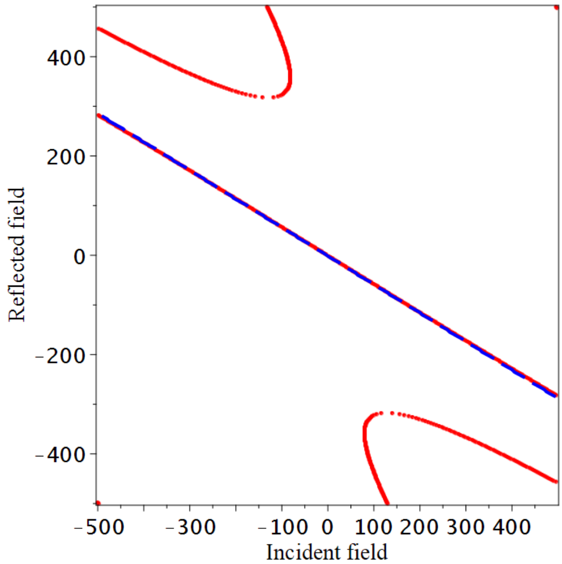



Рис. .4: Amplitude of the reflected field $F_R$ vs amplitude of the incident field $F_I$ in the linear ($\alpha = 0$, blue) and nonlinear (red) cases. The values of parameters are $\gamma = 1.15,\ k_0^2 = 1,\ r_0 \ = \ 2,\ r \ = \ 4,\ \varepsilon^2 = 4,\ \alpha = 10^{-3}$.

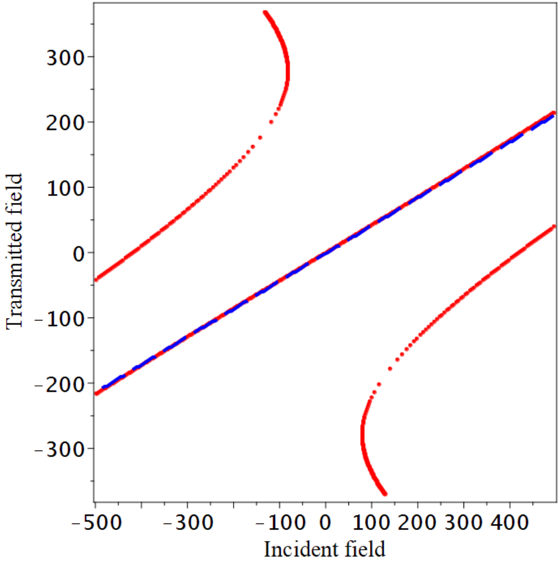



Рис. .5: Amplitude of the transmitted field $F_T$ vs amplitude of the incident field $F_I$ in the linear ($\alpha = 0$, blue) and nonlinear (red) cases. The values of parameters are$\gamma = 1.15,\ k_0^2 = 1,\ r_0 = 2,\ r = 4,\ \varepsilon^2 = 4,\ \alpha = 10^{-3}$.

Thus for fixed value of $F_I$, when the number $F_R = \widetilde{F}_R$ is such that $\Delta\left(F_I, F_R\right) = 0$, then $F_R$ is the solution of problem $P$ which corresponds to the value of $F_I$.
In Figs. 2–5 the amplitudes of the reflected, $F_R$ , and transmitted, $F_T$, fields calculated with respect to the amplitude of the incident field $F_I$ are shown.
These simulation results describe the essential relationships between linear and nonlinear problems. Namely, the nonlinear reflected field can be predicted from that obtained from the linear problem using the perturbation theory method (for small value of nonlinearity coefficient $\alpha$). Uniqueness of the solution to the nonlinear problem is preserved, see Figs. 2 and 3. Note that the curves in Figs. 4 and 5 significantly different from linear curves. Uniqueness of the solution to the nonlinear problem is not preserved for the "big" value of nonlinearity coefficient $\alpha$.

We have developed an analytical-numerical approach for the analysis of electromagnetic wave diffraction by a waveguide of circular geometry filled with nonlinear metamaterial medium. The method can be extended to more complicated nonlinearities and applied to numerical solution of the problems of diffraction by multilayered metal-dielectric structures with nonlinear media.

- R. W. P. King, Tai Tsun Wu, "The Scattering and Diffraction of Waves," *Harvard University Press, London, UK*, 1959.
- Y. Miyazaki, "Scattering and diffraction of electromagnetic waves by inhomogeneous dielectric cylinder," *Radio Science*, **16**, 1981, pp. 1009–1014.
- D.V. Valovik, E. Smolkin, "Calculation of the propagation constants of inhomogeneous nonlinear double-layer circular cylindrical waveguide by means of the Cauchy problem method," *Journal of Communications Technology and Electronics*, **58**, 2013, pp. 759–767.
- Yu.G. Smirnov, E. Smolkin, "On the existence of non-polarized azimuthal-symmetric electromagnetic waves in circular dielectric waveguide filled with nonlinear isotropic homogeneous medium," *Wave Motion*, **77**, 2018, pp. 77–90.
- E. Smolkin, Yu.V. Shestopalov, "Nonlinear Goubau line: analytical-numerical approaches and new propagation regimes," *Journal of Electromagnetic Waves and Applications*, **31(8)**, 2017, pp. 781–797.
- E. Smolkin, Y. Shestopalov, M. Snegur, "Diffraction of TM polarized electromagnetic waves by a nonlinear inhomogeneous metal-dielectric waveguide," *Proceedings of the 2017 19th International Conference on Electromagnetics in Advanced Applications, ICEAA 2017*, 8065508, 2017, pp. 1288–1291.
- M.J. Adams, "An Introduction to Optical Waveguide," *John Wiley and Sons, New York, NY, USA*, 1981.
- M. Abramowitz, I.A. Stegun, "Handbook of Mathematical Functions," *National Bureau of Standards, Washington, USA*, 1972.