WAVES IN A CLOSED REGULAR WAVEGUIDE OF ARBITRARY CROSS-SECTION

YURY SHESTOPALOV, EUGENE SMOLKIN AND MAXIM SNEGUR

Penza State University

Analysis of wave propagation in a regular waveguides with inhomogeneous filling and arbitrary inclusions (perfectly conducting) constitutes an important class of electromagnetic problems. However, many problems here remain unsolved, in particular, existence of normal waves and their basic properties including the discreteness and localization spectrum of normal waves on the complex plane and the completeness of system of eigenvectors and associated vectors.
The theory of electromagnetic wave propagation in regular waveguides with homogeneous filling were elaborated in classical works of Tikhonov and Samarskii [1,2]. For non-homogeneous waveguides with given cross-sectional geometry, in particular, rectangular and circular, the results concerning existence and distribution of the normal wave spectrum on the complex plane were obtained by reducing the original problem to explicit dispersion relations and analysis of the corresponding complex-valued functions of one or several complex variables.
An approach based on the reduction to eigenvalue problems for operator pencils considered in Sobolev spaces was proposed by Smirnov in [3,4]. General theory of polynomial operator-functions called operator pencils is sufficiently well elaborated. A fundamental work by Keldysh [5] pioneered investigation of non-self-adjoint polynomial pencils. For inhomogeneous waveguide structures of arbitrary cross section, an approach [6] based on reducing the problem to the study of an operator-function is proposed. Theorems are proved concerning the discrete character of the spectrum of the problem and the distribution of characteristic numbers of the operator function over the complex plane. Additionally, we prove that the system of eigen- and associated vectors of the operator function is doubly complete in the sense of Keldysh with a finite defect.

[1] A.A. Samarskii, A.N. Tikhonov, "On excitation of radio waveguides II," *Zhurnal Tekhnicheskoj Fiziki*, **17**, 1947, pp. 1431–1440.
[2] A.A. Samarskii, A.N. Tikhonov, "The representation of the field in waveguide in the form of the sum of TE and TM modes," *ZhurnalTekhnicheskoj Fiziki*, **18**, 1948, pp. 971–985.
[3] Yu.G. Smirnov, "Application of the operator pencil method in the eigenvalue problem for partially," *Doklady AN SSSR* **312**, 1990, pp. 597–599.
[4] Yu.G. Smirnov, "The method of operator pencils in the boundary transmission problems for elliptic system of equations," *Differential Equations*, **27**, 1991, pp. 140–147.
[5] M.V. Keldysh, "On the completeness of the eigenfunctions of some classes of non-selfadjoint linear operators," *Doklady AN SSSR*, **77**, 1951, pp. 11–4.
[6] Yu.G. Smirnov, Eu. Smolkin, "Discreteness of the spectrum in the problem on normal waves in an open inhomogeneous waveguide," *Differential Equations*, **53**, 2017, pp. 1262–1273.

Consider the three-dimensional space $\mathbb{R}^3$ with the cartesian coordinate system $Oxyz$. Let $\Omega \in \mathbb{R}^2 = \{z = 0\}$ is a bounded domain on the plane Oxy with boundaries Γ_1 and Γ_2 (see Fig. 1). We will consider the problem of normal waves in cylindrical shielded waveguide which transversal (with respect to Oz) crosssection is formed by the domain Ω. We assume that waveguide's filling contains isotropic inhomogeneous media with the relative dielectric permittivity $\varepsilon(x, y)$ and magnetic permeability $\mu(x, y)$. The boundaries Γ_1 and Γ_2 are the projection of the surface of the infinitely thin and perfectly conducting screens.

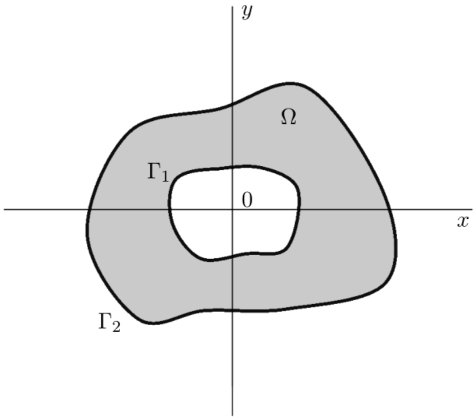

Рис. .1: Geometry of the problem

The permittivity and permeability inside a waveguide have a form $\varepsilon_0\varepsilon(\mathrm{x}),\ \mathrm{x}\in\overline{\Omega}$, and $\mu_0\mu(\mathrm{x}),\ \mathrm{x}\in\overline{\Omega}$,respectivelly, where $\mathrm{x}=(x,y)$, $\varepsilon(\mathrm{x})\in C^1(\overline{\Omega})$ and $\mu(\mathrm{x})\in C^1(\overline{\Omega})$. Here ε_0 and μ_0 are permittivity and permeability of vacuum.
We will consider monochromatic waves

$$\mathbf{E}e^{-i\omega t}=e^{-i\omega t}\left(\mathrm{E}_x,\mathrm{E}_y,\mathrm{E}_z\right)^T,\ \ \mathbf{H}e^{-i\omega t}=e^{-i\omega t}\left(\mathrm{H}_x,\mathrm{H}_y,\mathrm{H}_z\right)^T,$$

where $(\ \cdot\ )^T$ denotes the transpose operation. Each component of the field $\mathbf{E}$, $\mathbf{H}$ is a function of three spatial variables.
The complex amplitudes $\mathbf{E}$, $\mathbf{H}$ of the electromagnetic field satisfy Maxwell's equations

$$\begin{cases}\operatorname{rot}\mathbf{H}=-i\varepsilon\mathbf{E},\\ \operatorname{rot}\mathbf{E}=i\mu\mathbf{H}.\end{cases} \tag{1}$$

boundary conditions for tangential electric components on perfectly conducting surfaces:

$$\mathbf{E}_\tau|_{\Gamma_1}=0,\ \ \mathbf{E}_\tau|_{\Gamma_2}=0, \tag{2}$$

and the finite energy condition:

$$\int\limits_{\Omega}\left(\varepsilon|\mathbf{E}|^2+\mu|\mathbf{H}|^2\right)d\mathrm{x}<\infty, \tag{3}$$

where τ denotes the tangential unit vector.

The Maxwell system (1) is written in the normalized form. The passage to dimensionless variables has been carried out [7]; namely, $k_0\mathrm{x} \to \mathrm{x}$, $\gamma \to \frac{\gamma}{k_0}$, $\sqrt{\frac{\mu_0}{\varepsilon_0}}\mathbf{H} \to \mathbf{H}$, $\mathbf{E} \to \mathbf{E}$, where $k_0^2 = \omega^2\varepsilon_0\mu_0$ (the time factor $e^{-i\omega t}$ is omitted everywhere.)

The normal waves propagating along the axis Oz of the waveguide W have the form [8]

$$\begin{aligned} &\mathrm{E}_x = E_x(\mathrm{x})e^{i\gamma z}, \quad \mathrm{E}_y = E_y(\mathrm{x})e^{i\gamma z}, \quad \mathrm{E}_z = E_z(\mathrm{x})e^{i\gamma z}, \\ &\mathrm{H}_x = H_x(\mathrm{x})e^{i\gamma z}, \quad \mathrm{H}_y = H_y(\mathrm{x})e^{i\gamma z}, \quad \mathrm{H}_z = H_z(\mathrm{x})e^{i\gamma z}, \end{aligned} \tag{4}$$

where γ is the normalized propagation constant of waveguide (unknown spectral parameter of the problem). The problem (1)–(3) is an eigenvalue problem for the Maxwell equations with spectral parameter γ. In what follows we often omit the arguments of functions when it does not lead to misunderstanding.

[7] Yu.G. Smirnov, "Mathematical Methods for Electromagnetic Problems," *Penza: PSU Press*, 2009.
[8] A.W. Snyder, J. Love, "Optical waveguide theory," *Springer*, 1983. 1907.

Substituting $\mathbf{E}$ and $\mathbf{H}$ with components (4) into equations (1), expressing the functions $E_x,\ E_y,\ H_x$ and H_y through E_z and H_z, we find

$$E_x = \frac{i}{\kappa^2}\left(\gamma\frac{\partial E_z}{\partial x} + \mu\frac{\partial H_z}{\partial y}\right),\ H_x = \frac{i}{\kappa^2}\left(\gamma\frac{\partial H_z}{\partial x} - \varepsilon\frac{\partial E_z}{\partial y}\right),$$
$$E_y = \frac{i}{\kappa^2}\left(\gamma\frac{\partial E_z}{\partial y} - \mu\frac{\partial H_z}{\partial x}\right),\ H_y = \frac{i}{\kappa^2}\left(\gamma\frac{\partial H_z}{\partial y} + \varepsilon\frac{\partial E_z}{\partial x}\right),$$

where

$$\kappa^2 = \varepsilon\mu - \gamma^2 \neq 0.$$

It follows from last formulas that the field of the normal wave can be represented via two scalar functions

$$\Pi := E_z(\mathrm{x}),\ \Phi := H_z(\mathrm{x}).$$

For functions Π and Φ from (1)–(3) we have the following eigenvalue problem P: to find $\gamma \in \mathbb{C}$, called eigenvalues such that there are nontrivial solutions of the system

$$\begin{cases} \Delta\Pi + \kappa^2\Pi = \dfrac{\gamma^2}{\varepsilon\kappa^2}\nabla\varepsilon\nabla\Pi + \dfrac{\gamma}{\varepsilon\kappa^2}J\left(\varepsilon\mu, \Phi\right) + \dfrac{\gamma\varepsilon}{\kappa^2}\nabla\mu\nabla\Pi, \\ \Delta\Phi + \kappa^2\Phi = \dfrac{\gamma^2}{\mu\kappa^2}\nabla\mu\nabla\Phi + \dfrac{\gamma}{\mu\kappa^2}J\left(\varepsilon\mu, \Pi\right) + \dfrac{\gamma\mu}{\kappa^2}\nabla\varepsilon\nabla\Phi, \end{cases}$$

and

$$J\left(u, v\right) := \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x},$$

satisfying the boundary conditions on Γ_1 and Γ_2

$$\Pi|_{\Gamma_1} = 0, \ \left.\frac{\partial\Phi}{\partial n}\right|_{\Gamma_1} = 0, \ \Pi|_{\Gamma_2} = 0, \ \left.\frac{\partial\Phi}{\partial n}\right|_{\Gamma_2} = 0,$$

and the energy condition

$$\int\limits_{\Omega}\left(|\nabla\Pi|^2 + |\nabla\Phi|^2 + |\Pi|^2 + |\Phi|^2\right)d\mathrm{x} < \infty,$$

where n denotes the (exterior w.r.t. Ω) normal unit vector such that $x \times y = \tau \times n$.
The equivalence of reduction to the problem P is not valid only for $\gamma^2 = \varepsilon\mu$; in this case it is necessary to study the system (1) directly.

We will find the solutions Π and Φ of the problem P in Sobolev spaces: $H_0^1(\Omega)$ and $H^1(\Omega)$, respectively, with the inner product and the norm

$$(f,g)_1 = \int\limits_{\Omega} \left(\nabla f \nabla \overline{g} + f\overline{g}\right) dx, \quad \|f\|_1^2 = (f,f)_1 .$$

Let us give the variational formulation of the problem P. Multiplying the equations of system by arbitrary test functions $u \in H_0^1(\Omega)$, $v \in H^1(\Omega)$ (we can assume that these functions are continuously differentiable in $\overline{\Omega}$), and applying Green's formula [?], taking into account the boundary conditions and the right-hand sides of the equations of the system under consideration, we obtain we obtain the *variational relation*

$$\begin{aligned}\gamma^2 \int\limits_{\Omega} (\Pi\overline{u} + \Phi\overline{v})\, dx + \int\limits_{\Omega} (\nabla\Pi\nabla\overline{u} + \nabla\Phi\nabla\overline{v})\, dx - \\ - \int\limits_{\Omega} \varepsilon\mu\, (\Pi\overline{u} + \Phi\overline{v})\, dx + \int\limits_{\Omega} \frac{\gamma^2}{\kappa^2}\left(\frac{\nabla\varepsilon\nabla\Pi}{\varepsilon}\overline{u} + \frac{\nabla\mu\nabla\Phi}{\mu}\overline{v}\right) dx + \\ + \int\limits_{\Omega} \frac{\gamma}{\kappa^2}\left(\frac{J(\varepsilon\mu,\Phi)}{\varepsilon}\overline{u} + \frac{J(\varepsilon\mu,\Pi)}{\mu}\overline{v}\right) dx + \\ + \int\limits_{\Omega} \frac{\gamma}{\kappa^2}\left(\varepsilon\overline{u}\nabla\mu\nabla\Pi + \mu\overline{v}\nabla\varepsilon\nabla\Phi\right) dx = 0, \quad (5)\end{aligned}$$

for all $u \in H_0^1(\Omega)$, $v \in H^1(\Omega)$.

Let $H = H_0^1(\Omega) \times H^1(\Omega)$ be the Cartesian product of the Hilbert spaces with the inner product and the norm

$$(\mathbf{u}, \mathbf{v}) = (u_1,\ v_1)_1 + (u_2,\ v_2)_1,\ \|\mathbf{u}\|^2 = \|u_1\|_1^2 + \|u_2\|_1^2,$$

where $\mathbf{u} = (u_1,\ u_2)^T,\ \mathbf{v} = (v_1,\ v_2)^T$.
The integrals in (5) can be considered as the sesquilinear forms on $\mathbb{C}$, defined in H with respect to vector-functions

$$\mathbf{u} = (\Pi,\ \Phi)^T,\ \mathbf{v} = (u,\ v)^T.$$

These forms (if they are bounded) define, in accordance with the results of [9], linear bounded operators $\mathrm{T} : H \to H$

$$\mathrm{t}(\mathbf{u},\ \mathbf{v}) = (\mathrm{T}\mathbf{u},\ \mathbf{v}),\ \forall \mathbf{v} \in H, \tag{6}$$

[9] T. Kato, "Perturbation Theory for Linear Operators," *New York: Springer-Verlag*, 1980.

Let us consider the following quadratic forms and corresponding operators

$$\mathrm{k}(\mathbf{u},\ \mathbf{v}) := \int\limits_{\Omega} (\Pi\overline{u} + \Phi\overline{v})\, d\mathrm{x} = (\mathrm{K}\mathbf{u}, \mathbf{v}),$$

$$\mathrm{k}_1(\mathbf{u},\ \mathbf{v}) := \int\limits_{\Omega} (\varepsilon\mu + 1)\,(\Pi\overline{u} + \Phi\overline{v})\, d\mathrm{x} = (\mathrm{K}_1\mathbf{u}, \mathbf{v}),$$

$$\mathrm{a}(\mathbf{u},\ \mathbf{v}) := \int\limits_{\Omega} \Big(\nabla\Pi\nabla\overline{u} + \nabla\Phi\nabla\overline{v} + \varepsilon\Pi\overline{u} + \Phi\overline{v}\Big) d\mathrm{x} = (\mathrm{I}\mathbf{u}, \mathbf{v}),$$

$$\mathrm{b}_1(\mathbf{u},\ \mathbf{v}) := \int\limits_{\Omega} \frac{\gamma^2}{\kappa^2}\left(\frac{\nabla\varepsilon\nabla\Pi}{\varepsilon}\overline{u} + \frac{\nabla\mu\nabla\Phi}{\mu}\overline{v}\right) d\mathrm{x} = (\mathrm{B}_1(\gamma)\mathbf{u}, \mathbf{v}),$$

$$\mathrm{b}_2(\mathbf{u},\ \mathbf{v}) := \int\limits_{\Omega} \frac{\gamma}{\kappa^2}\left(\frac{J\,(\varepsilon\mu, \Phi)}{\varepsilon}\overline{u} + \frac{J\,(\varepsilon\mu, \Pi)}{\mu}\overline{v}\right) d\mathrm{x} = (\mathrm{B}_2(\gamma)\mathbf{u}, \mathbf{v}),$$

$$\mathrm{b}_3(\mathbf{u},\ \mathbf{v}) := \int\limits_{\Omega} \frac{\gamma}{\kappa^2}\,(\varepsilon\overline{u}\nabla\mu\nabla\Pi + \mu\overline{v}\nabla\varepsilon\nabla\Phi)\, d\mathrm{x} = (\mathrm{B}_3(\gamma)\mathbf{u}, \mathbf{v}),$$

for all $\mathbf{v} \in H$.

The variational problem (5) can be written in the operator form

$$\mathrm{N}(\gamma)\mathbf{u} := \left(\gamma^2\mathrm{K} + \mathrm{I} - \mathrm{K}_1 + \mathrm{B}_1(\gamma) + \mathrm{B}_2(\gamma) + \mathrm{B}_3(\gamma)\right)\mathbf{u} = 0.$$

The characteristic numbers and eigenvectors of the operator-function $\mathrm{N}(\gamma)$ by definition coincide with the eigenvalues and eigenvectors of the problem P for $\gamma^2 \neq \mu\varepsilon$.
Thus the problem of normal waves is reduced to the eigenvalue problem for the operator-function $\mathrm{N}(\gamma)$. In this way we consider properties of the operators in (11).

Lemma 1

The operator K *is compact. The following estimate holds for its eigenvalues*

$$\lambda_n(\mathrm{K}) = O(n^{-1}), \quad n \to \infty.$$

Lemma 2

The operator-function $\mathrm{B}_1(\gamma)$, $\mathrm{B}_2(\gamma)$ *and* $\mathrm{B}_3(\gamma)$ *are compact (bounded) and holomorphic in the domain*

$$\mathbb{C}\backslash\Lambda_0, \textit{ where } \Lambda_0 := \{\gamma : \ \gamma^2 = \mu(\mathrm{x})\varepsilon(\mathrm{x}), \ \mathrm{x} \in \overline{\Omega}\}.$$

Definition 3

We will denote by $\rho(\mathrm{N})$ the resolvent set of $\mathrm{N}(\gamma)$ (consisting of all values of $\gamma \in \mathbb{C}$ where there exists the bounded inverse operator $\mathrm{N}^{-1}(\gamma)$) and by $\sigma(\mathrm{N}) = \mathbb{C} \setminus \rho(\mathrm{N})$ the spectrum of $\mathrm{N}(\gamma)$.

Properties of the spectrum of the operator-function $\mathrm{N}(\gamma)$ are given in the following theorems.

Theorem 4

There exists $\widetilde{\gamma} \in \mathbb{R}$ *such that the operator* $\mathrm{N}(\widetilde{\gamma})$ *is continuously invertible, i.e. resolvent set* $\rho(\mathrm{N}) := \{\gamma : \exists\, \mathrm{N}^{-1}(\gamma) : H \to H\}$ *of operator-function* $\mathrm{N}(\widetilde{\gamma})$ *is not empty.*

Theorem 5

Operator $\mathrm{N}(\gamma) : H \to H$ *is bounded, holomorphic, and Fredholm in the domain* $\Lambda = \mathbb{C} \backslash \Lambda_0$.

Theorem 6

The spectrum of operator-function $\mathrm{N}(\gamma) : H \to H$ *is discrete in* Λ *i.e. has a finite number of eigenvalues of finite algebraic multiplicity in any compact set* $\mathrm{K}_0 \subset \Lambda$.

Lemma 7

The spectrum of the operator-function $\mathrm{N}(\gamma)$ *is symmetric with respect to the origin* $\sigma(\mathrm{N}) = -\sigma(\mathrm{N})$. *If* γ_0 *is eigenvalue of operator-function* $\mathrm{N}(\gamma)$ *corresponding to eigenvector* $\mathbf{u} = (\Pi, \Phi)^T$ *then value* $-\gamma_0$ *is also eigenvalue of operator-function* $\mathrm{N}(\gamma)$ *corresponding to eigenvector* $\mathbf{u} = (-\Pi, \Phi)^T$ *with the same multiplicity.*

Let us consider operator-function $\mathrm{N}(\gamma)$ in domain $\Lambda_\eta := \{\gamma : |\gamma| > \eta\}$, where η is arbitrary positive value such that $\eta > \sqrt{\max\limits_{\mathrm{x} \in \overline{\Omega}} |\mu(\mathrm{x})\varepsilon(\mathrm{x})|}$.

Theorem 8

System of eigenvectors and associated vectors of the operator-function $\mathrm{N}(\gamma)$ *corresponding to eigenvalues located in domain* Λ_η *is double complete with a finite defect in* $H \times H$*:*

$$\dim \operatorname{coker} \overline{\widetilde{\mathrm{N}}\left(\phi_{\mathrm{p}}^{(\mathrm{k},0)}\right)} < \infty \text{ and } \dim \operatorname{coker} \overline{\widetilde{\mathrm{N}}\left(\phi_{\mathrm{p}}^{(\mathrm{k},1)}\right)} < \infty;$$

where $\overline{\widetilde{\mathrm{N}}\left(\phi_p^{(k,\nu)}\right)}$ *denotes the closure of linear combinations of vectors* $\left\{\phi_p^{(k,\nu)}\right\}$.

We have reduced the boundary eigenvalue problem for the Maxwell equations describing normal waves in a broad class of non-homogeneously filled waveguides to an eigenvalue problem for an operator-function. We have proved fundamental properties of the spectrum of normal waves including the discreteness and a statement describing localization of eigenvalues of the operator-function on the complex plane.
We have formulated the definition of eigenwaves and associated waves of a waveguide in terms of eigenvectors and associated vectors of an operator-function. We have established double completeness of the system of eigenvectors and associated vectors of the operator-function with a finite defect. We have proved the existence of an infinite (countable) set of eigenvalues located in domain $\Lambda_\eta := \{\gamma : |\gamma| > \eta\}$.

- A.A. Samarskii, A.N. Tikhonov, "On excitation of radio waveguides II," *Zhurnal Tekhnicheskoj Fiziki*, **17**, 1947, pp. 1431–1440.
- A.A. Samarskii, A.N. Tikhonov, "The representation of the field in waveguide in the form of the sum of TE and TM modes," *ZhurnalTekhnicheskoj Fiziki*, **18**, 1948, pp. 971–985.
- Yu.G. Smirnov, "Application of the operator pencil method in the eigenvalue problem for partially," *Doklady AN SSSR* **312**, 1990, pp. 597–599.
- Yu.G. Smirnov, E. Smolkin, M.O. Snegur, "Analysis of the Spectrum of Azimuthally Symmetric Waves of an Open Inhomogeneous Anisotropic Waveguide with Longitudinal Magnetization," *Computational Mathematics and Mathematical Physics*, **58(11)**, 2018, pp. 1887–1901.
- Yu.G. Smirnov, "The method of operator pencils in the boundary transmission problems for elliptic system of equations," *Differential Equations*, **27**, 1991, pp. 140–147.
- M.V. Keldysh, "On the completeness of the eigenfunctions of some classes of non-selfadjoint linear operators," *Doklady AN SSSR*, **77**, 1951, pp. 11–4.
- T. Kato, "Perturbation Theory for Linear Operators," *New York: Springer-Verlag*, 1980.
- R. Adams, "Sobolev spaces," *New York: Academic Press*, 1975.
- I.C. Gohberg, M.G. Krein, "Introduction to the Theory of Linear Nonselfadjoint Operators in Hilbert Space," *M.: Nauka*, 1965.
- Yu.G. Smirnov, Eu. Smolkin, "Discreteness of the spectrum in the problem on normal waves in an open inhomogeneous waveguide," *Differential Equations*, **53**, 2017, pp. 1262–1273.