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Phaseless near-field techniques from a random starting point

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U.R.S.I.

The phase retrieval problem

Phase retrieval consists in the reconstruction of a signal from only intensity data

Applications in Electromagnetics

- Array diagnostics or antenna diagnostics
- Reconstruction of radiated fields
- Inverse scattering problem

Mathematical formulation

Phase retrieval is a quadratic inverse problem that can be formulated as recovering the unknown vector $\underline{x} \in C^N$ from the set of equations

$$B(\underline{x}) = \underline{M}^2$$

Quadratic operator that links the unknowns to the data

$$B: \underline{x} \in C^N \rightarrow \left|\underline{\underline{L}}\,\underline{x}\right|^2 \in R^M$$

$$\text{where } \left|\underline{\underline{L}}\,\underline{x}\right|^2 = \left(\underline{\underline{L}}\,\underline{x}\right) \odot \left(\underline{\underline{L}}\,\underline{x}\right)^*$$

Phaseless Data

$$\underline{M}^2 = \left[M_1^2, M_2^2, \dots, M_M^2\right]^T \in R^M$$

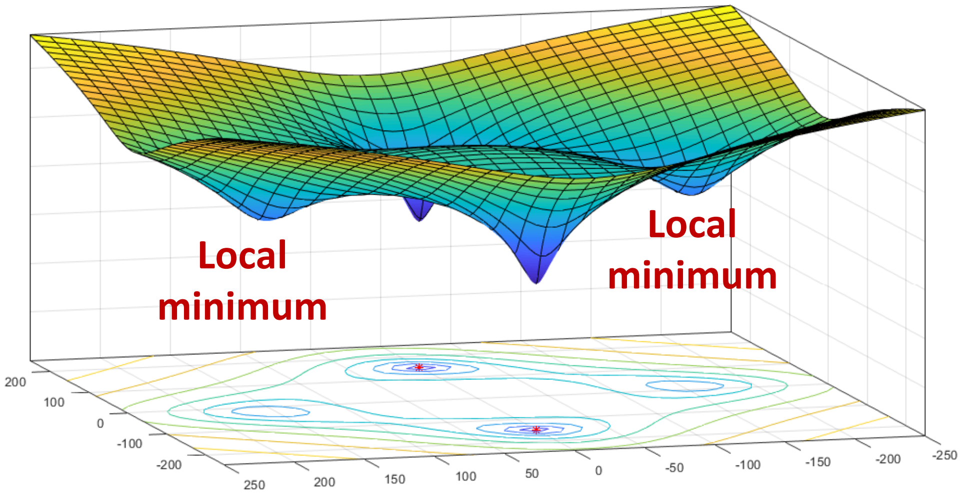
The question of local minima

Quadratic inversion method

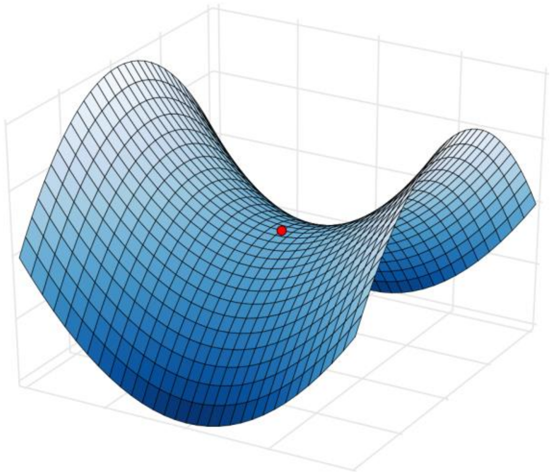
$$\min_{\underline{x} \in C^N} \phi\left(\underline{x}\right) \quad \text{where} \quad \phi\left(\underline{x}\right) = \left\| B\left(\underline{x}\right) - \underline{M}^2 \right\|_2^2$$

Possible presence of <u>TRAP POINTS</u>

Local minima



Saddle points with null gradient



The evaluation of the dimension of data

The dimension of data can be defined as the number of independent functions required to represent the data with a given degree of accuracy.

How is it possible to evaluate the dimension of data space ?

1. By introducing a linear representation of the data through a redefinition of the unknown space
2. By evaluating the number of relevant singular values of the linear operator that represents the data

$B(\underline{x}) = \underline{M}^2$ → $A(\underline{X}) = \underline{M}^2$

System of M quadratic equations in N unknowns

$\underline{x} = (\mathrm{x}_1, \mathrm{x}_2, \dots, \mathrm{x}_\mathrm{N})$

System of M linear equations in N^2 unknowns

$\underline{X}$ is made up by all the products $x_i x_j^*$

The dimension of data space M_{ind} is equal to number of relevant singular values of A

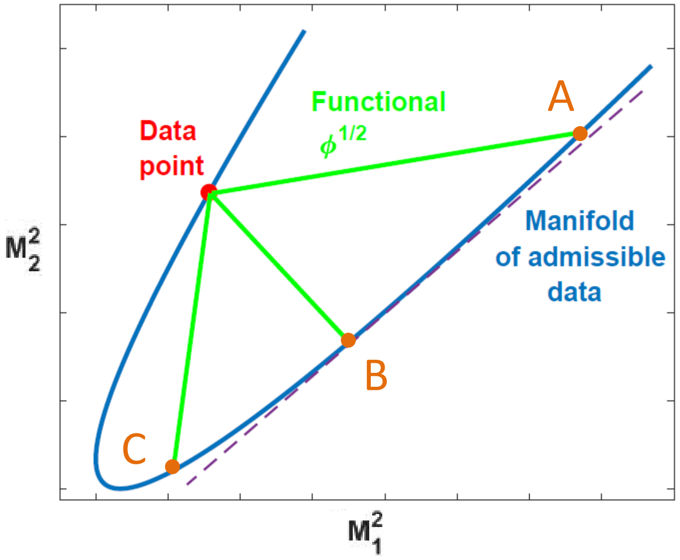
The origin of local minima

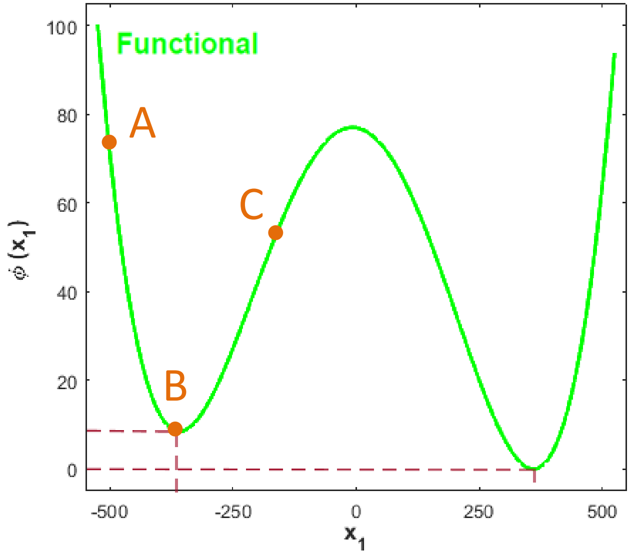
The image of the quadratic operator $B(\underline{x})$ represents the manifold of data

The data vector $\underline{M}^2$ is a point of the manifold of data (in absence of noise)

The functional ϕ is a measure in the data space of the distance between the point $B(\underline{x})$ and the data point $\underline{M}^2$

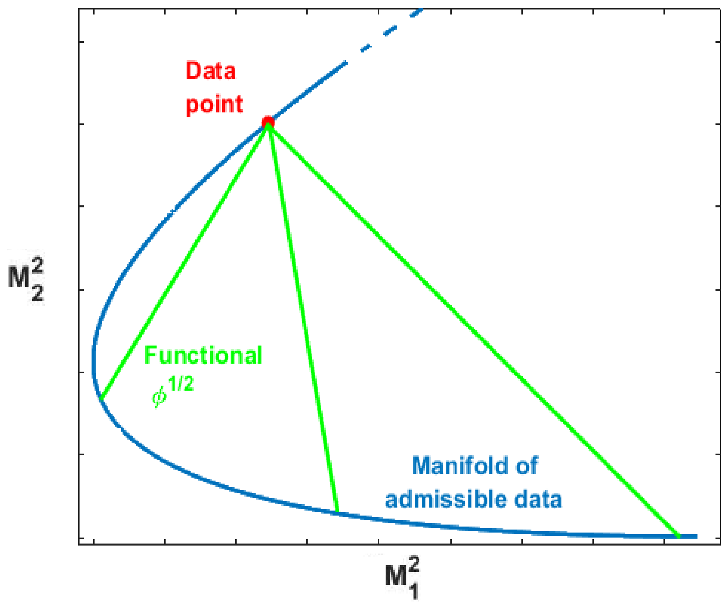
<u>**Local minima arise since the manifold of data has a non zero curvature**</u>



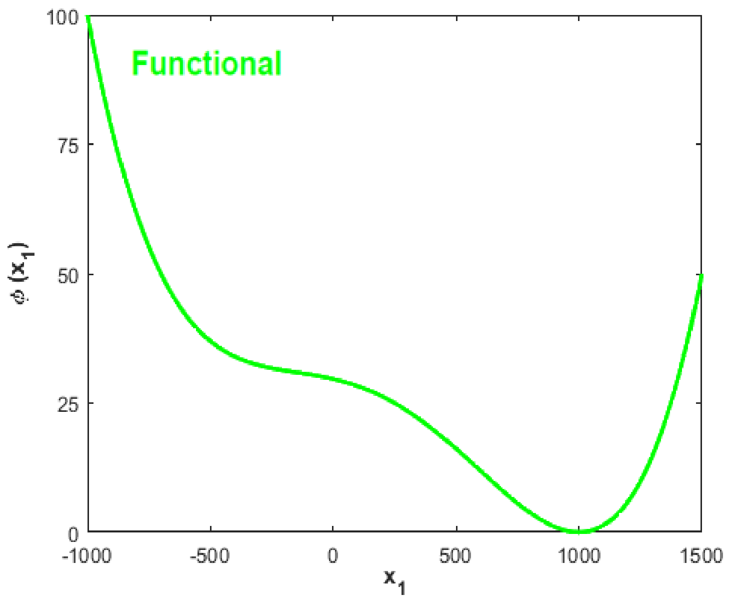


Example for a simple case of 1 real unknown and 2 data

How to “cure” local minima



Manifold of data large enough



No local minima in the functional

If the ratio $\dfrac{\textit{dimension of data space } (M_{ind})}{\textit{number of unknonws } (N_r)}$ is high enough,

the manifold of data is sufficiently large and no trap points appear in the functional.

Strategies to solve local minima question

How to cure local minima?

By increasing the dimension of data

In which way is it possibile to increase M_{ind} ?

By increasing of the number of scanning surface

By dividing the numerical procedure in more steps,
and by searching in each step only a reduced numer of unkowns

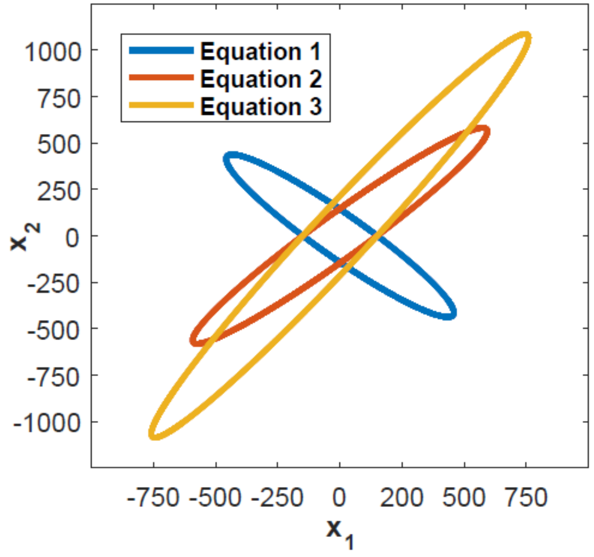
When is this possible ?

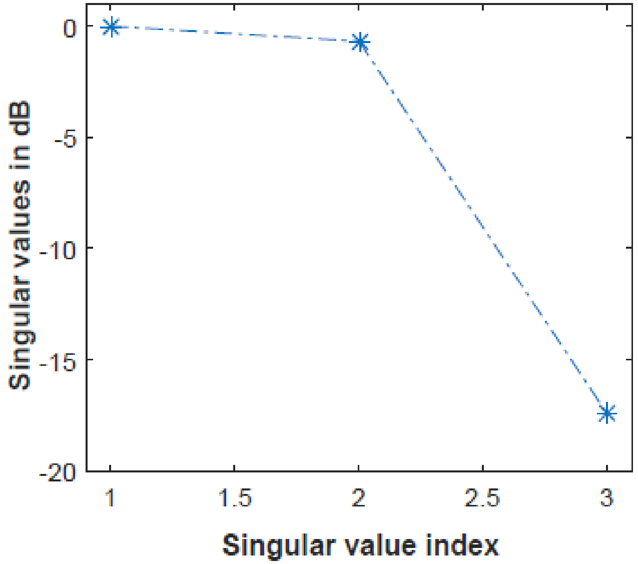
This is possible if
1. some components of the unknown vector $\underline{x}$ are zero or quasi - zero
2. the location of such zeros is known or it can be estimated from data

Curing local minima by increasing the data (1)

Diagrams of the equations

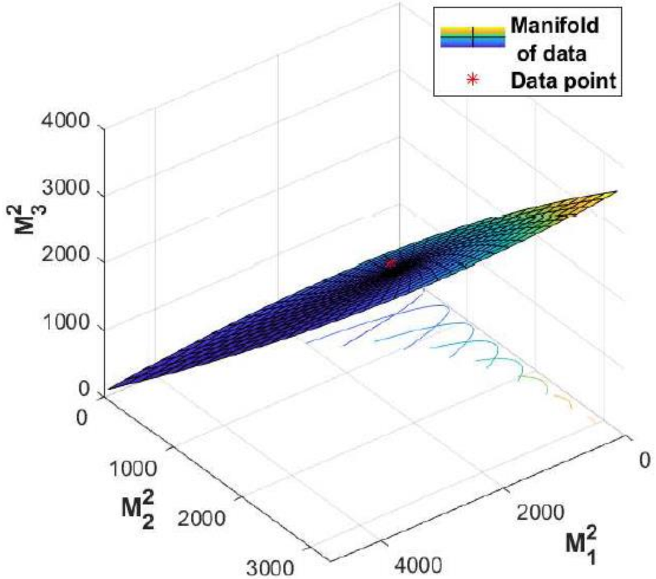
Equations 2 and 3 are quite similar

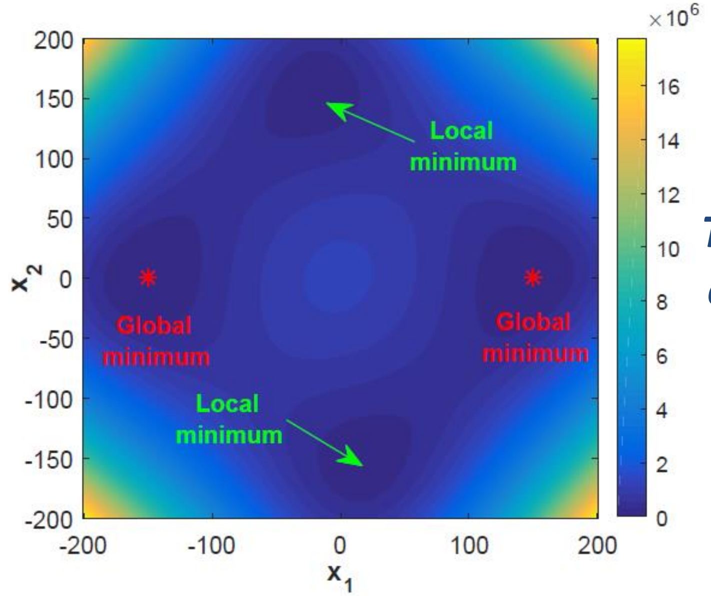




Singular values of the lifting operator

Manifold of data





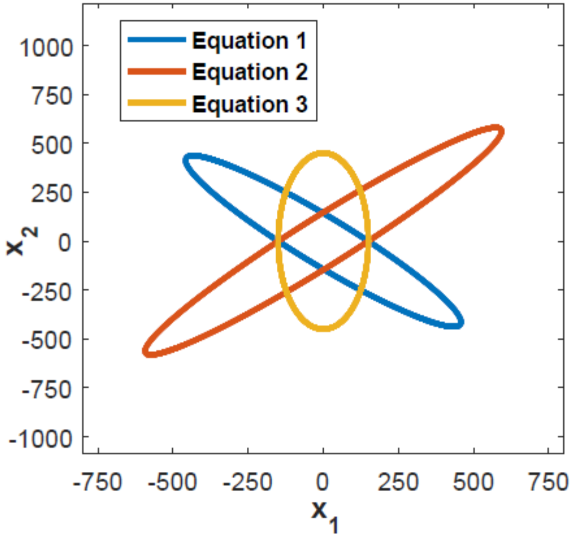
Quartic functional

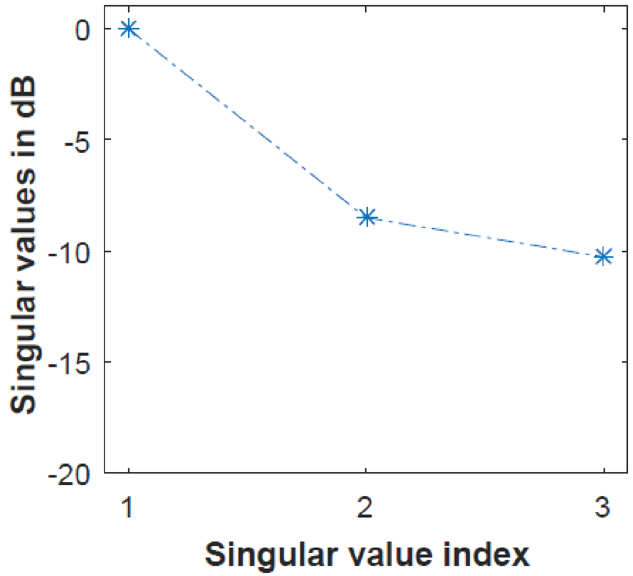
The functional contains local minima

Curing local minima by increasing the data (2)

Diagrams of the equations

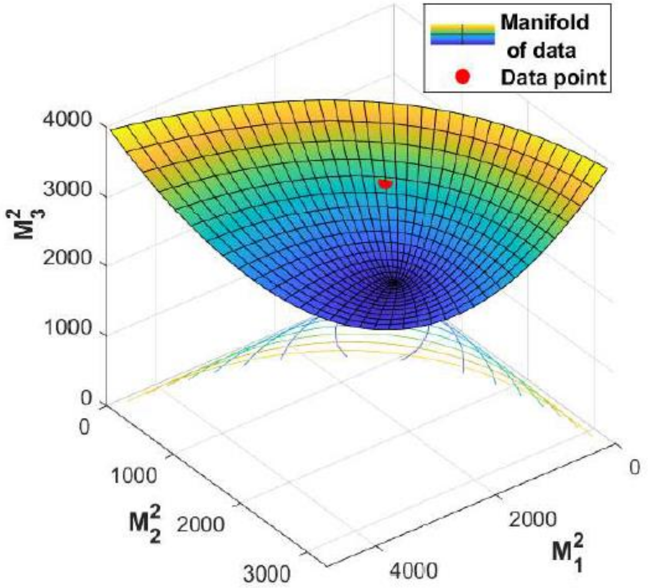
All the 3 equations are different each other

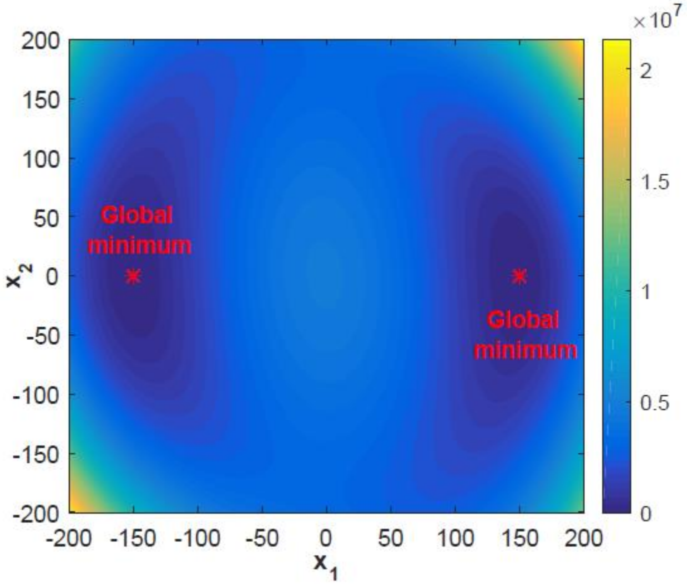




Singular values of the lifting operator

Manifold of data





Quartic functional

Local minima disappear

Curing local minima by searching less unknowns

Numerical procedure to escape from local minima

First step of the procedure

- Localizing the zero or quasi zero components of $\underline{x} \in C^N$, and creating the vector $\underline{x}_{rel} \in C^{N_1}$ by deleting from $\underline{x}$ all the zero and quasi-zero components.

- Since the zero components of $\underline{x}$ does not contribute to the data $\underline{M}^2$, instead, to solve the system $\left|\underline{\underline{L}}\,\underline{x}\right|^2 = \underline{M}^2$ it is possible to solve the problem

$$\left|\underline{\underline{L}}_{rel}\,\underline{x}_{rel}\right|^2 = \underline{M}^2$$

where $\underline{\underline{L}}_{rel}$ is the matrix obtained by deleting the columns of $\underline{\underline{L}}$ corresponding to the zero components of $\underline{x}$. The solution of the quadratic system above can be found by minimizing

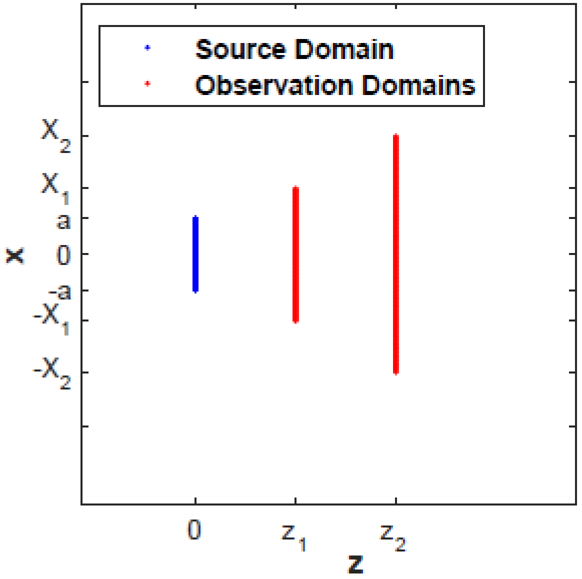
$$\phi(\underline{x}_{rel}) = \left\|\left|\underline{\underline{L}}_{rel}\,\underline{x}_{rel}\right|^2 - \underline{M}^2\right\|^2$$

Second step of the procedure

- Minimizing the functional $\phi(\underline{x}) = \left\|\left|\underline{\underline{L}}\,\underline{x}\right|^2 - \underline{M}^2\right\|^2$ starting from an initial guess whose main components are equal to $\underline{x}_{rel}$ and zero otherwise. *This starting point is in the attraction region !!!*

Geometry of the test cases

2D scalar radiation problem



The data are collected 2 lines circles at points (z_i , x_m)

The number of phaseless measurements collected on the p−th circle are $M_p = 4N_0 + 1$ with $N_0 = \frac{\beta a}{\pi}$

Mathematical model

$$| E(z_i , x_m) |^2 \approx \left| \sum_{n=-N_0}^{N_0} I(u_n)\, \psi_n(z_i, x_m) \right|^2 \quad \begin{array}{l} i = 1, \dots , N_c \\ m = 1, \dots , M_p \end{array}$$

Samples of the pseudo sampling functions

Unknowns (Samples of the radiation integral)

$$\underline{x} = [x_1 , \dots , x_N]^T = [I(u_{-N_0}) , \dots , I(u_{N_0})]^T$$

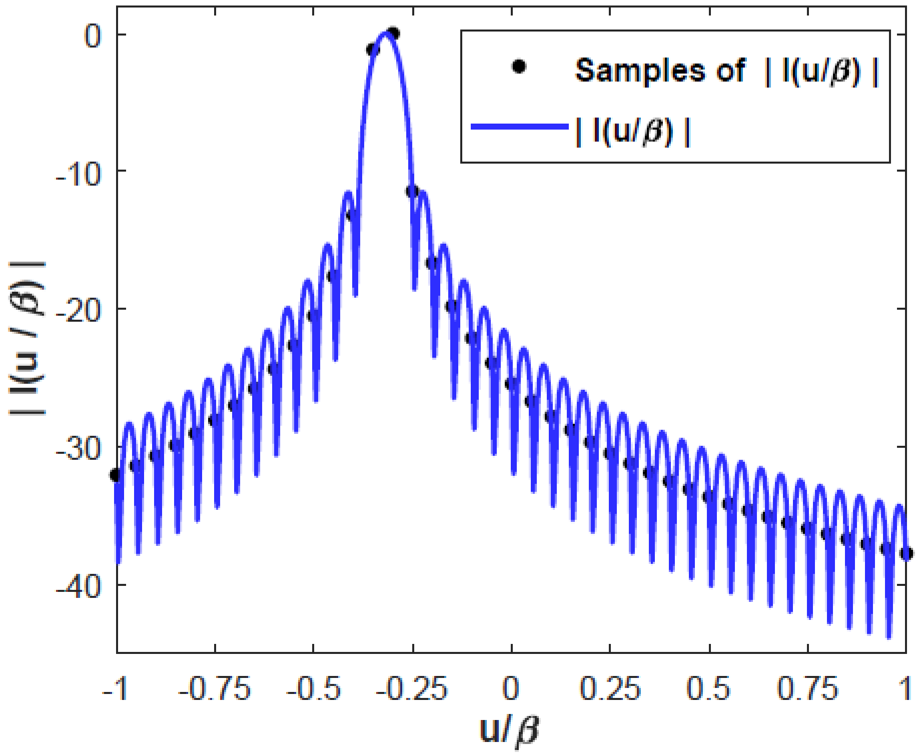
with $N = 2N_0 + 1$

Phaseless Data

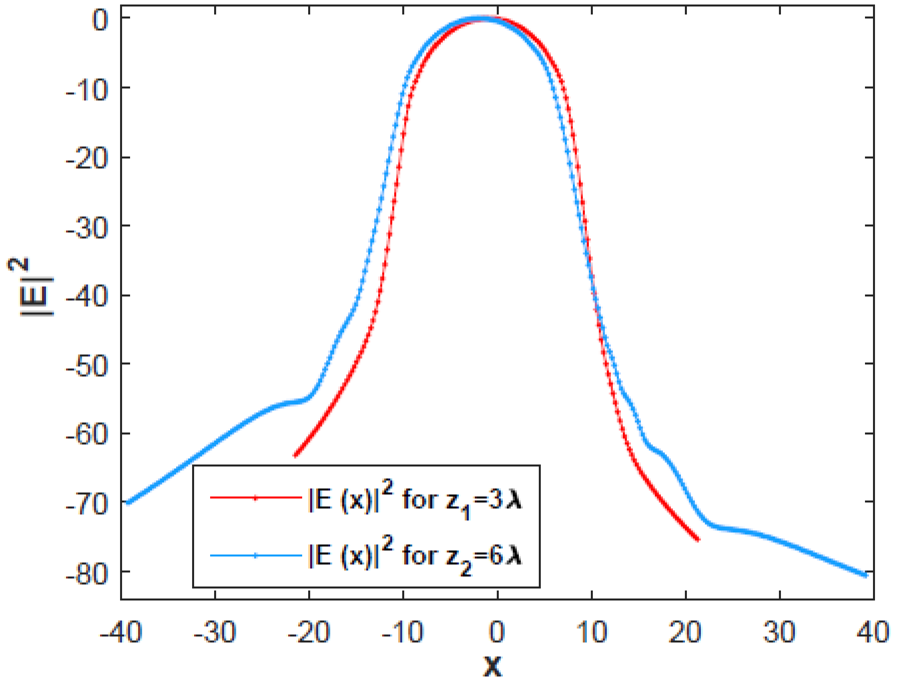
$$\underline{M}^2 = \left[| E(z_1 , x_1) |^2, | E(z_1 , x_2) |^2, \dots , \left| E\left(z_2 , x_{M_p}\right) \right|^2 \right]$$

Unknown sequence, and data

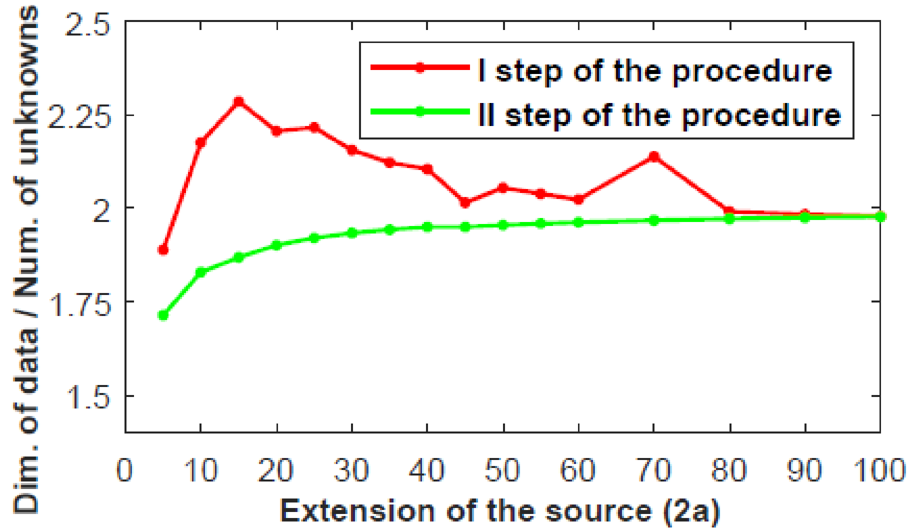
Unknown sequence (Samples of the radiation integral I(u)



Square amplitude of the radiated field in dB on different lines parallel to the source

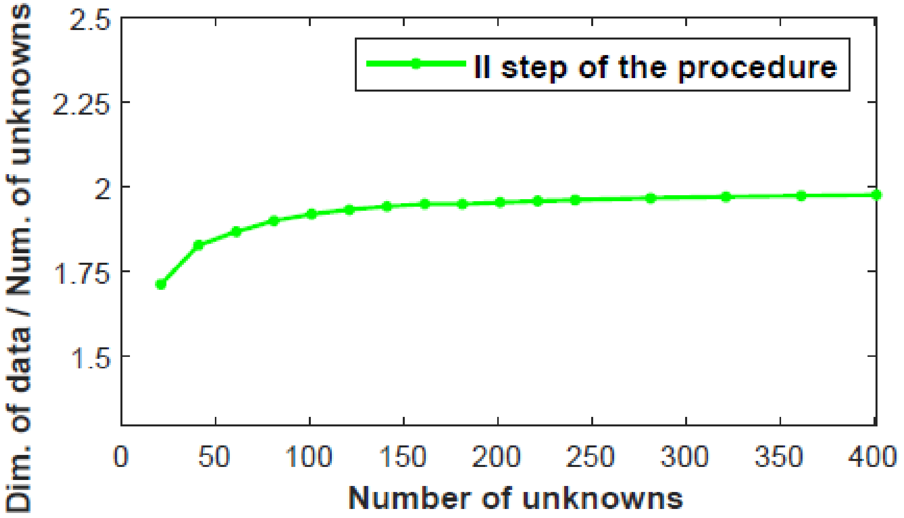


Results of the numerical analysis



Comparison of the ratio $\frac{\textit{dimension of data space}}{\textit{number of unknonws}}$

in the first and in the second step of the procedure.



Value of the ratio $\frac{\textit{dimension of data space}}{\textit{number of unknonws}}$

for which the numerical procedure converges.

The proposed method compared with the classical quadratic inversion method requires a lower number of data. Since it reconstructs N unknowns with the number of data required to reconstruct the N_1 non zero unknowns.