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An asymptotic evaluation of a kernel in the study of a radiation operator: the strip current in near zone

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The SVD in linear inverse problems

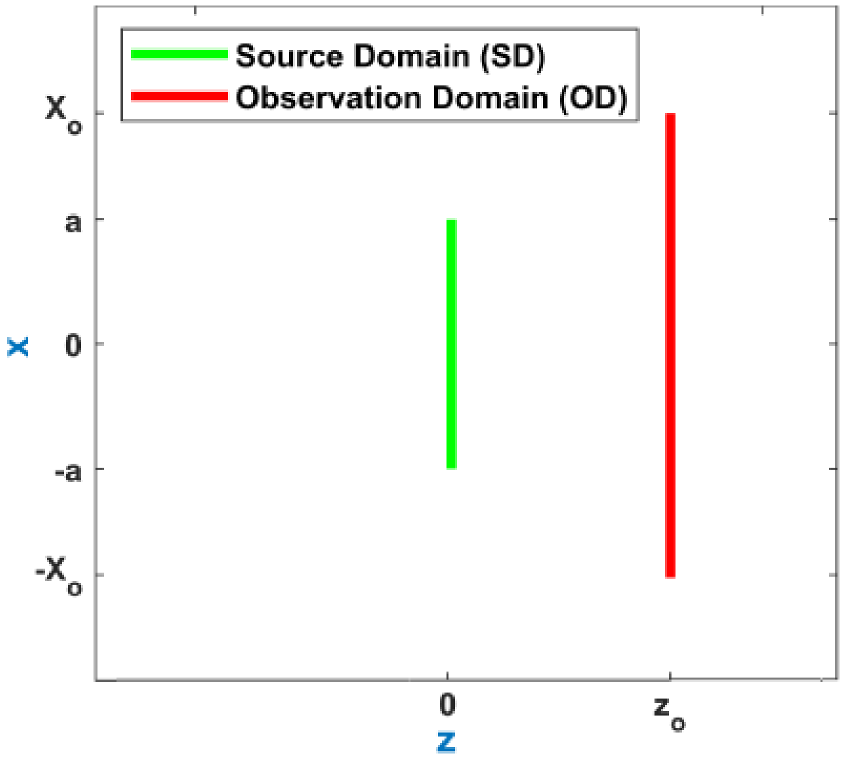
The singular values decomposition (SVD) of an operator represents a key mathematical tool in linear inverse problems.

The relevance of the singular system is due essentially to the following two reasons:

1. it allows expressing the unknown function of the linear problem explicitly.
2. the singular system is related to some figures of merit like the number of degrees of freedom (NDF), and the point spread function (PSF).

Unfortunately, in some cases the singular system of the relevant operator is not known in closed-form; in such cases, numerical computation of the singular system must be performed.
Despite this, the numerical computation of the eigenspectrum does not allow to show the role played by the geometry, and by the configuration parameters; for this reason, a closed-form expression of the singular system is desirable.

Geometry of the problem



Geometry of the problem

Radiation operator

$$TJ = \int_{-a}^{a} g(x_o, x')\ J_m(x')\ dx'$$

Green function

$$g(x_o, x') = \frac{z_o}{R^{\frac{3}{2}}(x_o, x')} e^{-j\beta\, R(x_o, x')}$$

where $\quad R(x_o, x') = \sqrt{(x_o - x')^2 + z_o^2}$

Study of the radiation operator

In order to study the mathematical properties of the radiation operator, we will refer to the integral operator TT^H.

$$TT^H E = \int_{-X_o}^{X_o} K(x, x_o)\, E(x_o)\, dx_o$$

The kernel $K(x, x_o)$ can be expressed as below

$$K(x, x_o) = \int_{-a}^{a} f(x', x, x_o)\, e^{-j\,\beta a\,\phi(x', x, x_o)}\, dx'$$

where

$$f(x', x, x_o) = \frac{1}{R^{\frac{3}{2}}(x, x')R^{\frac{3}{2}}(x_o, x')} \qquad \phi(x', x, x_o) = \frac{R(x, x') - R(x_o, x')}{a}$$

Asymptotic evaluation of the kernel

For βa >> 1 the kernel $K(x, x_o)$ can be evaluated by exploiting an asymptotic approach which provides the following approximation of the kernel

$$K(x, x_o) \approx -\frac{z_o^2}{j\beta a}\left(\frac{f(a, x, x_o)}{\phi'(a, x, x_o)}\, e^{-j\,\beta a\,\phi(a, x, x_o)} - \frac{f(-a, x, x_o)}{\phi'(-a, x, x_o)}\, e^{-j\,\beta a\,\phi(-a, x, x_o)}\right)$$

The kernel of $T\,T^H$ is non-convolution and non-bandlimited !!!

The kernel in new variables (η, η_o)

Thanks to the introduction of the following variables

$$\eta_o = \frac{1}{2a}\left(\sqrt{(x_o + a)^2 + z_o^2} - \sqrt{(x_o - a)^2 + z_o^2}\right)$$

$$\gamma_o = \frac{1}{2a}\left(\sqrt{(x_o + a)^2 + z_o^2} + \sqrt{(x_o - a)^2 + z_o^2}\right)$$

it is possible to recast the operator TT^H <u>in a form more similar to a convolution operator of difference type.</u>

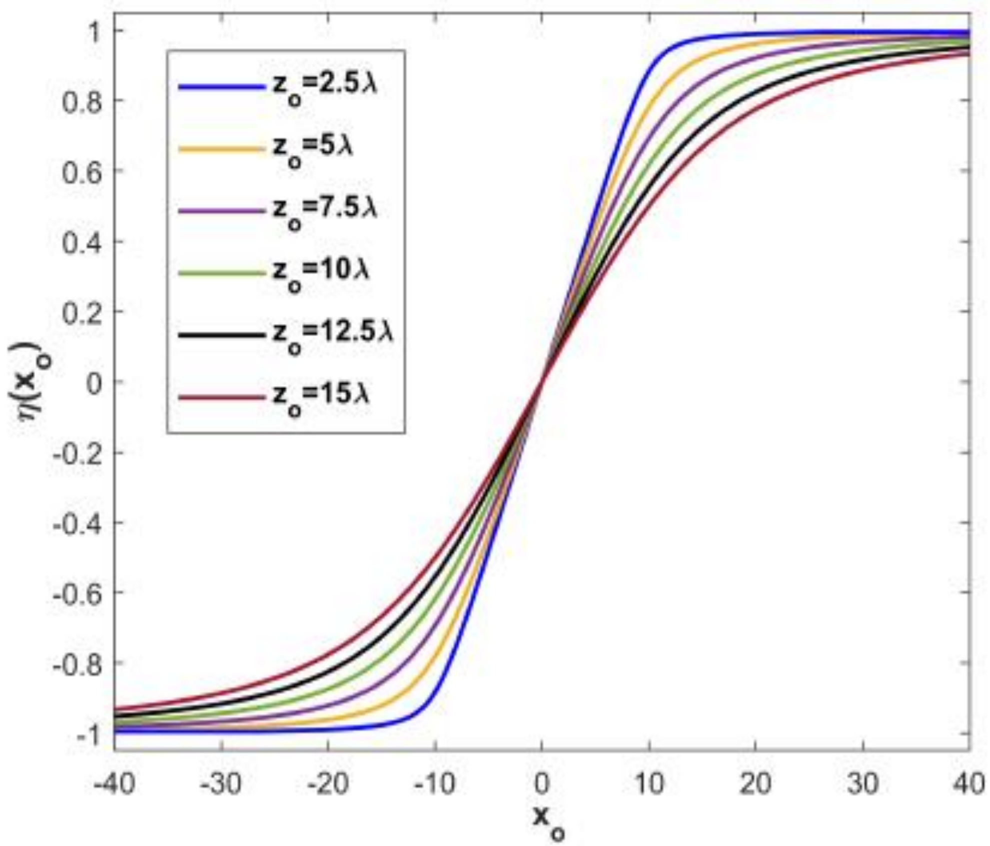
The introduction of the variables above, linked with some approximations of the amplitude terms allow recasting the kernel of TT^H in the form

$$K(\eta, \eta_o) \approx e^{-j\beta a(\gamma(\eta)-\gamma(\eta_o))}\,\frac{\sin\big(\beta a(\eta - \eta_o)\big)}{\pi(\eta - \eta_o)}$$

(convolution, bandlimited kernel of sinc type)

The behavior of the transformation

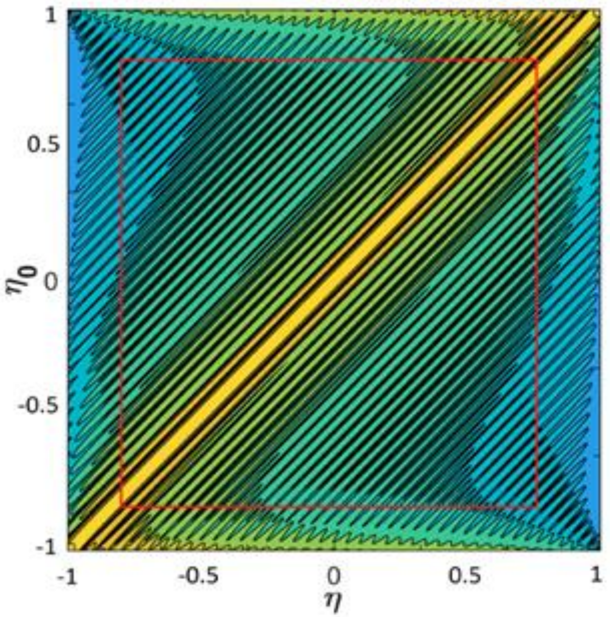
$$\eta\,(x_o) = \frac{1}{2a}\left(\sqrt{(x_o + a)^2 + z_o^2} - \sqrt{(x_o - a)^2 + z_o^2}\right)$$



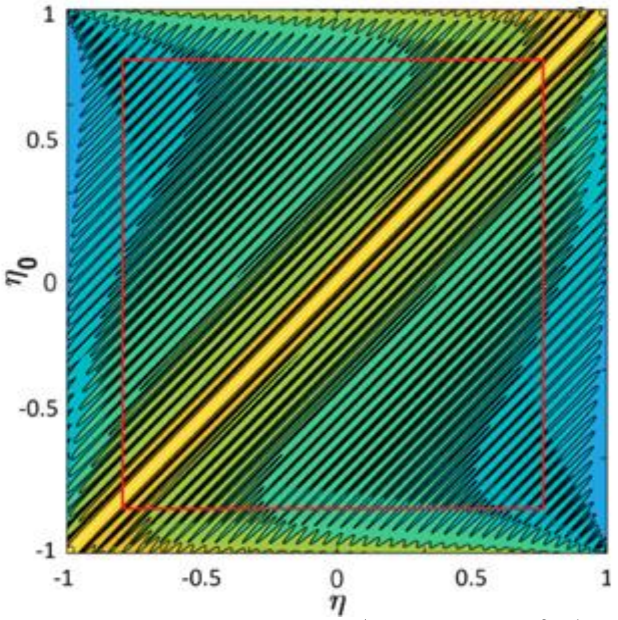
Diagrams of $\eta\,(x_o)$ *for different values of* z_o *when* $a = 10\lambda$

Diagrams of the kernel and its approximations

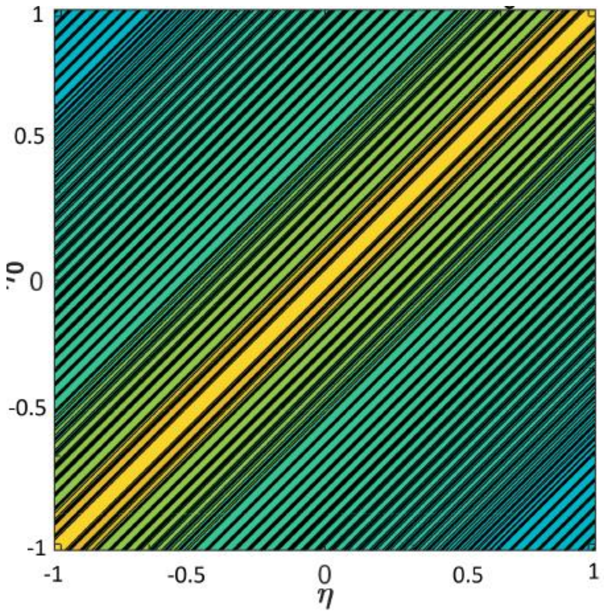
The three figures refers to the configuration $a = 10\lambda$ $(\eta(a) = 0.78)$, $z_o = 5\lambda$



Actual kernel in the variables (η, η_o)



Asymptotic evaluation of the kernel in the variables (η, η_o)



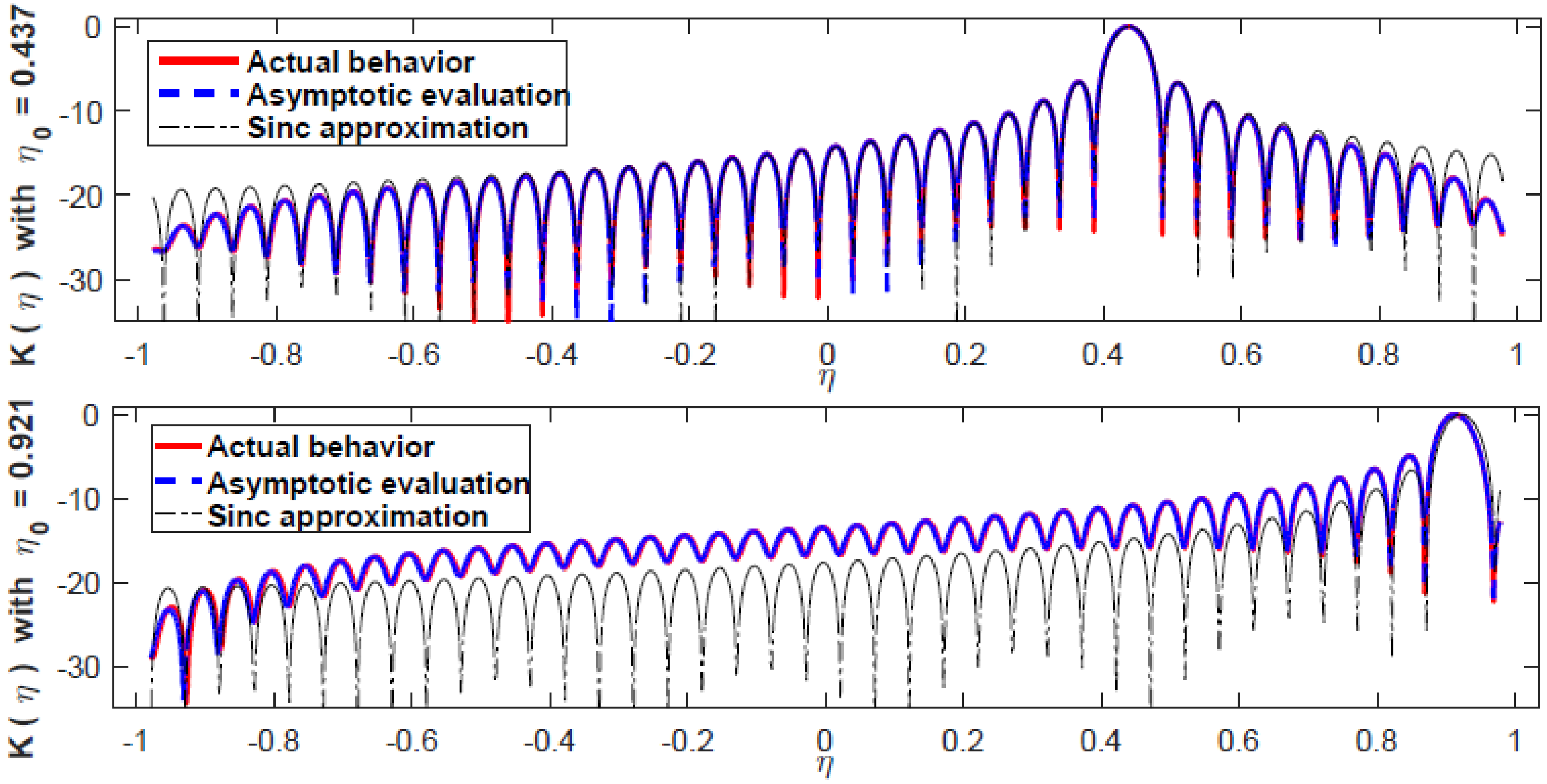
Sinc approximation of the kernel

The sinc kernel represents a good approximation of the actual kernel

$$\forall(\boldsymbol{\eta}, \boldsymbol{\eta_o}) \in [\boldsymbol{\eta}(-\boldsymbol{a}), \boldsymbol{\eta}(\boldsymbol{a})] \times [\boldsymbol{\eta}(-\boldsymbol{a}), \boldsymbol{\eta}(\boldsymbol{a})]$$

Hence, the sinc approximation of the kernel works until the extension of the observation domain is less or equal than the extension of the source.

Some cuts of the kernel



$\boldsymbol{\eta_o} \in [\boldsymbol{\eta}(-\boldsymbol{a}), \boldsymbol{\eta}(\boldsymbol{a})]$

The sinc approximation overlaps with the actual kernel $\forall \boldsymbol{\eta} \in [\boldsymbol{\eta}(-\boldsymbol{a}), \boldsymbol{\eta}(\boldsymbol{a})]$

$\boldsymbol{\eta_o} \notin [\boldsymbol{\eta}(-\boldsymbol{a}), \boldsymbol{\eta}(\boldsymbol{a})]$

The sinc approximation does not overlap with the actual kernel

The figures refers to the configuration $a = 10\lambda \; (\eta(a) = 0.78), \; z_o = 5\lambda$

Singular system of the radiation operator

Hence, for $X_o \leq a$

$$TT^H E \approx \int_{\eta(-X_o)}^{\eta(X_o)} e^{-j\beta a(\gamma(\eta)-\gamma(\eta_o))} \frac{\sin(\beta a(\eta-\eta_o))}{\pi(\eta-\eta_o)} E(\eta_o)\, d\eta_o$$

The eigenspectrum of such operator can be computed in closed-form by resorting to the Slepian Pollak theory.

Eigenvalues of TT^H

The eigenvalues of TT^H (that are the square of the singular values of the T) have a step-like behaviour with the knee occurring at the index

$$N = \frac{2}{\pi}\beta a\, \eta(X_o)$$

Eigenspectrum of TT^H

The eigenfunctions of TT^H (that are also the left singular functions of the T) are given by

$$v_n(\eta_o) = \frac{\psi_n(\eta_o, c)}{\sqrt{\lambda_n}} e^{j\beta a \gamma(\eta_o)}$$

where λ_n and ψ_n denote respectively the eigenvalues of the Slepian Pollak operator, and the prolate spheroidal waves functions.