Quasi-Classic Approximation for High Frequency Wave Field through Transionospheric Stochastic Channel of Propagation

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Introduction. Characterization of HF stochastic transionospheric channel of propagation

While propagating through the ionosphere with electron density irregularities, transionospheric radio signals may experience fluctuations. The moments of the random components of the field are commonly employed to characterize its stochastic properties.

The technique of Markov parabolic equations for the moments of stochastic field is one of the classical methods applied to solve the problems of wave propagation in random media that have been analyzed in a number of studies. The Markov parabolic equations allow for employing the quasi-classic approximation, which was first introduced in our papers [1, 2] for the case of the two-frequency two-position coherence function. This technique permits constructing an asymptotic solution for an arbitrary analytic model of the medium fluctuations.

In this paper, the numerical technique based on the quasi-classic approximation with complex-valued ray paths is presented for solving the Markov parabolic equation for the symmetric second order two-frequency and two-position coherence function of the high-frequency field propagating through a stochastic transionospheric channel.

1. A. A. Bitjukov, V. E. Gherm, N. N. Zernov, “On the solution of Markov’s parabolic equation for the second order spaced frequency and position coherence function,” *Radio Science*, **37**, 4, 2002, pp. 1-9, RS1066, doi: 10.1029/2001RS002491.
2. 2. A. A. Bitjukov, V. E. Gherm, N. N. Zernov, “Quasi-classic approximation in Markov’s parabolic equation for spaced position and frequency coherency,” *Radio Science*, **38**, 2, 2003, pp. 1-6, doi: 10.1029/2002RS002714.

Markov equation for coherence function

Equation for the symmetric second order spaced position and frequency coherence function $\Gamma_2(\boldsymbol{\rho}, z)$ of the complex amplitudes of the monochromatic waves propagating through the fluctuating medium with non-homogeneous background along the line of sight as follows

$$\frac{\partial \Gamma_2}{\partial z} + \frac{ik_d}{2k_1 k_2} \nabla_T^2 \Gamma_2 + \frac{k_d^2}{8} A_\varepsilon(0, z)\Gamma_2 + \frac{k_1 k_2}{8} D_\varepsilon(\boldsymbol{\rho}, z)\Gamma_2 = 0. \qquad (1)$$

Here $A_\varepsilon(\boldsymbol{\rho}, z)$and $D_\varepsilon(\boldsymbol{\rho}, z)$ are the effective transversal to the line of sight correlation and structure functions of the fluctuations of dielectric permittivity; $\boldsymbol{\rho}$ is the 2D difference spatial variable in the transversal plane, z is the coordinate along the line of sight; k_1 and k_2 are the wavenumbers, and $k_d = k_1 - k_2$. It is also assumed that the derivatives in the transversal central variables, defined by the characteristic scales of the background ionosphere, are negligibly small.

New unknown function $F(\boldsymbol{\rho}, z)$ is introduced according to the following relation:

$$\Gamma_2(\boldsymbol{\rho}, z) = F(\boldsymbol{\rho}, z) exp\left[-\frac{k_d^2}{8}\int_0^z A_\varepsilon(0, z')dz'\right]$$

So that the final equation for further investigation is:

$$\frac{\partial F}{\partial z} + \frac{ik_d}{2k_1 k_2} \nabla_T^2 F + \frac{k_1 k_2}{8} D_\varepsilon(\boldsymbol{\rho}, z)F = 0 \qquad (2)$$

Complex valued quasi-classic equations

The formal transition from equation (2) to its quasi-classic approximation is performed by tending the product $k_1 k_2$ to the infinity: $k_1 k_2 \to \infty$. The solution to the equation (2) is sought for in the form of an asymptotic series expansion

$$F(\boldsymbol{\rho}, z) = exp\big(k_1 k_2 \psi(\boldsymbol{\rho}, z)\big) \sum_{j=0}^{\infty} \frac{U_j(\boldsymbol{\rho}, z)}{(k_1 k_2)^j}$$

The main term of the solution to equation (2)

$$F(\boldsymbol{\rho}, z) \approx exp\big(k_1 k_2 \psi(\boldsymbol{\rho}, z)\big) U_0(\boldsymbol{\rho}, z). \qquad (3)$$

The complex-valued phase function $\psi(\boldsymbol{\rho}, z)$ (complex eikonal) is governed by the eikonal equation

$$\frac{\partial \psi}{\partial z} + \frac{i k_d}{2} (\nabla_T \psi)^2 + \frac{1}{8} D_\varepsilon(\boldsymbol{\rho}, z) = 0 \qquad (4)$$

The zero order amplitude U_0 obeys the main transport equation

$$\frac{\partial U_0}{\partial z} + i k_d (\nabla_T \psi \cdot \nabla_T U_0) + \frac{i k_d}{2} U_0 \nabla_T^2 \psi = 0. \quad (5)$$

Hamilton equations for the complex-valued eikonal equation

The general method of characteristics applied to the eikonal equation (4) produces the appropriate Hamilton equations

$$\frac{dz}{d\tau} = 1 \qquad \frac{dp_z}{d\tau} = -\frac{1}{8}\frac{\partial}{\partial z} D_\varepsilon(\boldsymbol{r}, z) \qquad \frac{d\boldsymbol{r}}{d\tau} = ik_d\boldsymbol{p} \qquad \frac{d\boldsymbol{p}}{d\tau} = -\frac{1}{8}\nabla_T D_\varepsilon(\boldsymbol{r}, z) \qquad (6)$$

Equations (6) determine complex-valued trajectories $\boldsymbol{r} = \boldsymbol{r}(\tau), 0 \le \tau \le z,$ which start at complex-valued points $(0, \boldsymbol{\rho}_0)$ on the initial surface $z = 0$, where the boundary (initial) condition is stated, and arrive at the real-valued points of observation $(z, \boldsymbol{\rho})$ and are subject to the initial conditions (in the case of plane wave $\Gamma = 1$ on the initial surface, hence $\boldsymbol{p}(0) = 0$, $p_z(0) = -\frac{1}{8} D_\varepsilon(\boldsymbol{\rho}_0, 0)$ and trajectories are orthogonal to the initial surface). The starting points and trajectories are determined for each point of observation and for each value of the frequency difference k_d.

Once the complex trajectories have been constructed, the complex eikonal ψ and the main amplitude U_0 are found as the appropriate integrals along the corresponding complex valued trajectories

$$\psi(\boldsymbol{\rho}, z) = \int_0^z \left(ik_d\boldsymbol{p}^2(\tau) + p_z(\tau)\right)d\tau, \qquad U_0(\boldsymbol{\rho}, z) = exp\left[-\frac{ik_d}{2}\int_0^z \nabla_T^2\psi(\boldsymbol{r}, \tau)d\tau\right] = \left[\frac{D(\boldsymbol{\rho}_0, 0)}{D(\boldsymbol{\rho}, z)}\right]^{1/2}, D(\boldsymbol{\rho}, z) = det\left\|\frac{\partial\boldsymbol{\rho}}{\partial\boldsymbol{\rho}_0}\right\|. \qquad (7)$$

In the limiting case of a homogeneous background medium and quadratic structure function of the dielectric permittivity fluctuations, the technique produces the rigorous well known solution to the spaced position and frequency coherence function.

Models of fluctuations and background transionospheric channel of propagation

The effective transversal structure function of the dielectric permittivity $D_\varepsilon(\boldsymbol{\rho}, z)$ is expressed through the effective transversal correlation function of the fractional electron density fluctuations $A_N(\boldsymbol{\rho})$ as follows

$$D_\varepsilon(\boldsymbol{\rho}, z) = 2\big(A_N(0) - A_N(\boldsymbol{\rho})\big)\frac{k_{pl}^4(z)}{k_1^2 k_2^2} \qquad k_{pl}^2(z) = \frac{e^2 n(z)}{\varepsilon_0 m_e c^2} \qquad N(\boldsymbol{\rho}, z) = \frac{\delta n(\boldsymbol{\rho}, z)}{n(z)}$$

Fractional electron density fluctuation

Assuming the anisotropic single-slope power law spectrum of fluctuations of the electron density in the magnetized ionosphere, we have for the arbitrary mutual orientations of the magnetic field and line of sight

$$A_N(\boldsymbol{\rho}) = \frac{a\sigma_N^2\, l_\perp}{\sqrt{2\pi}2^{((p-5)/2)}\Gamma((p-3)/2)\beta} \cdot \left(\frac{2\pi}{l_\perp}\sqrt{x^2 + \frac{y^2}{\beta^2}}\right)^{\frac{p-2}{2}} K_{\frac{p-2}{2}}\left(\frac{2\pi}{l_\perp}\sqrt{x^2 + \frac{y^2}{\beta^2}}\right). \qquad (8)$$

Here $a = l_\parallel / l_\perp$ is the aspect ratio of irregularities with $l_\parallel$ and $l_\perp$ being the outer scales of turbulence along and across the magnetic field respectively, σ_N^2 is the variance of the fractional electron density fluctuation, p is the 3D spectral index, $\beta^2 = cos^2(\varphi) + a^2 sin^2(\varphi)$ with φ being the angle between magnetic field and the direction of propagation, $K_\nu(\cdot)$ is the modified Bessel function of the second kind (Macdonald function). In this notation, x and y are the Cartesian components of the transversal vector variable $\boldsymbol{\rho}$, x is directed orthogonal to the magnetic field.

Models of fluctuations and background transionospheric channel of propagation

Equation (8) is valid for arbitrary anisotropy ($a = l_{\parallel}/l_{\perp}$) and orientation of the path of propagation (φ). The problem is generally 3D in space. However, there are limiting cases when the problem may be reduced to the 2D one.

1. Extremely anisotropic irregularities, propagation across the magnetic field $a \to \infty, \varphi = \pi/2$. Typical for the equatorial region. D_ε and, hence, solution do not depend on y. The problem is solved in variables (z, x).
2. Isotropic irregularities $a = 1$, arbitrary direction of propagation. D_ε and solution depend on $r = \sqrt{x^2 + y^2}$. The problem is solved in variables (z, r).

As a background ionospheric electron density profile in the simulations, ***Chapman layer model*** is utilized with the height of maximum of the electron density of 350 km, the characteristic scale of the layer of h_m=100 km, and critical frequency $f_{pl\,max}$ =10 MHz. The corresponding total electron content (TEC) is 49.7 TECu.

Anisotropy of the spatial coherence function

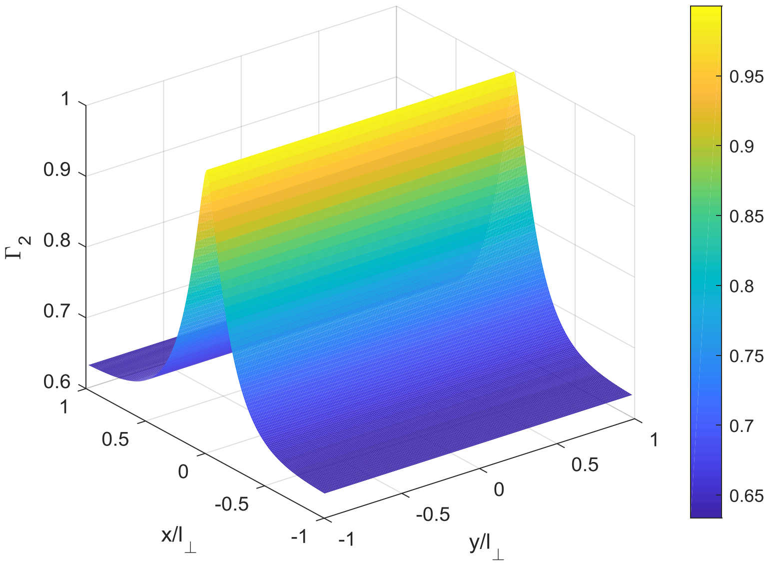

Figure 1. 3D plot on the ground of the coherence function for $k_d = 0$ and for the case of extremely anisotropic fluctuations $a \to \infty$, propagation across the magnetic field $\varphi = \pi/2$. Transmission frequency f_c=1 GHz, $l_\perp$=10 km, σ_N =0.025

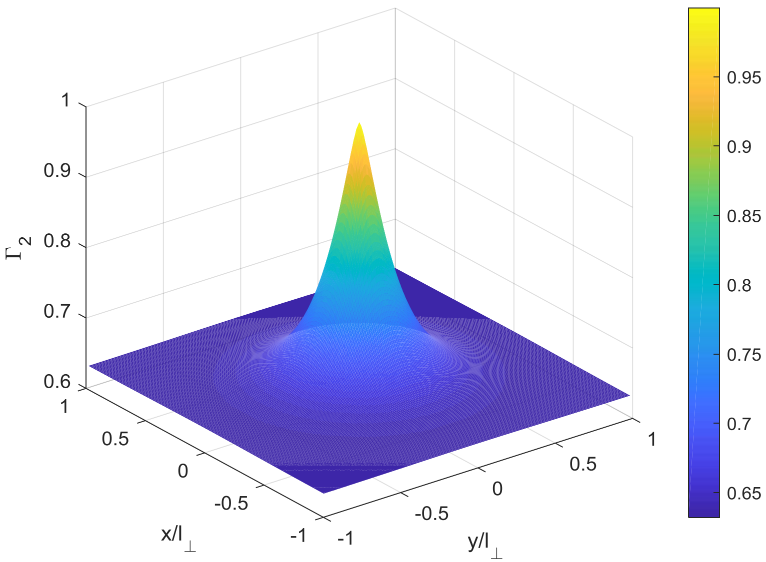

Figure 2. 3D plot on the ground of the coherence function for the case of isotropic fluctuations $a = 1$. Oher parameters are the same as in Fig. 1.

This is to demonstrate the anisotropy of the spatial coherence in case of anisotropic fluctuations (Fig.1) as compared to the isotropic case (Fig.2). In both cases the maximum coherence equal unity, then, expectedly, the coherence decays as the separation increases, tending to the constant value corresponding to the energy of the coherent component of a random field.

Coherence function in space-frequency domain

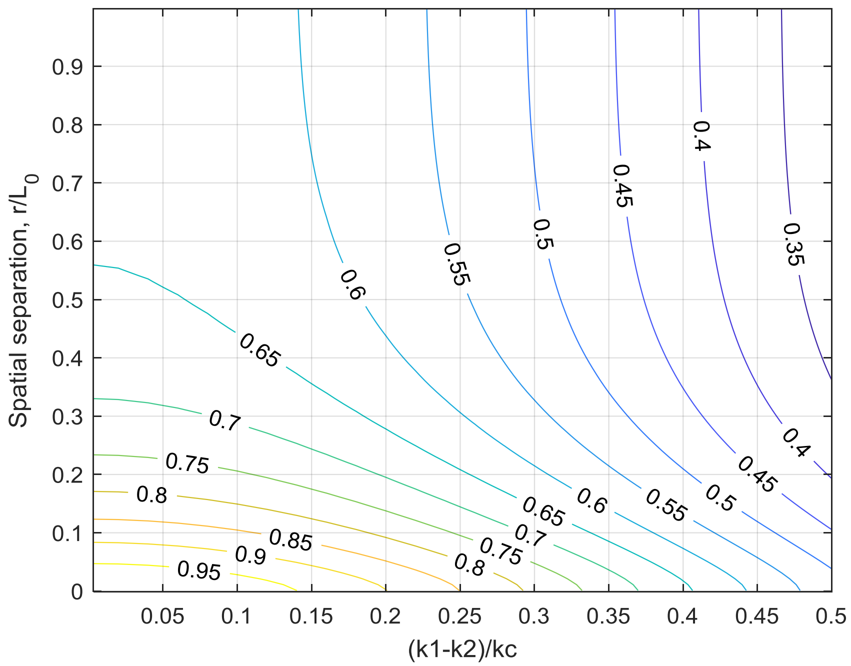

Figure 3. Contour plot of the absolute value of the coherence function in the domain $(k_d/k_c,\ r/l_\perp)$. f_c=1 GHz, $l_\perp$=10 km, σ_N =0.025

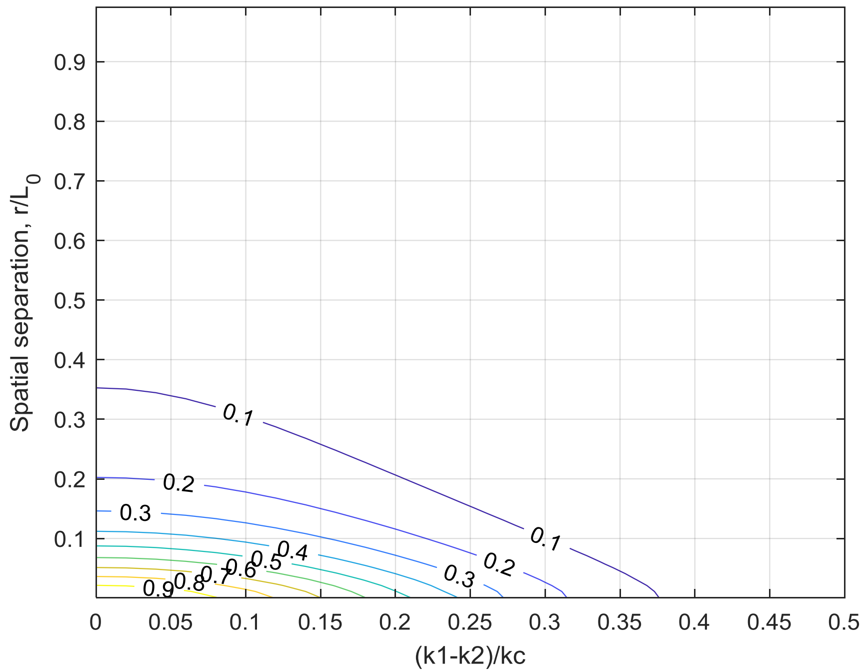

Figure 4. Contour plot of the absolute value of the coherence function in the domain $(k_d/k_c, r/l_\perp)$. f_c=400 MHz, $l_\perp$=10 km, σ_N =0.025

Unlike in spatial domain (Figs 1, 2, $k_d = 0$), where the coherence tends to the constant value, in the frequency domain it decays down to the small values as the frequency separation rises. The frequency bandwidth determined on the level 0.5 appears to be about 0.5 for the central transmission frequency 1 GHz (Fig. 3) and reduces to about 0.2 for the lower central frequency 400 MHz (Fig. 4). In the last case the coherence tends to zero in both spatial and frequency domains

Conclusions

- The earlier introduced quasi-classic technique for solving Markov equations for the statistical moments of the high-frequency random field, propagating through the stochastic transionospheric channel was presented and further discussed.
- Employing this technique, the effects of the background ionosphere and anisotropic local random inhomogeneities of the ionospheric electron density on the field coherence properties were presented and discussed.
- The quasi-classic approximation may serve as a good tool for verifying and validating different pure analytic or semi-analytic techniques for solving the same problem