Mutual Validation of a Fast Solver Based on the Multilevel Nonuniform Grid Approach and an Asymptotic Approximation for High-Frequency Scattering by Strongly Elongated Spheroids

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URSI GASS 2020

Outline

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- The MLNG-Based Iterative Solver
- Asymptotic Approximation
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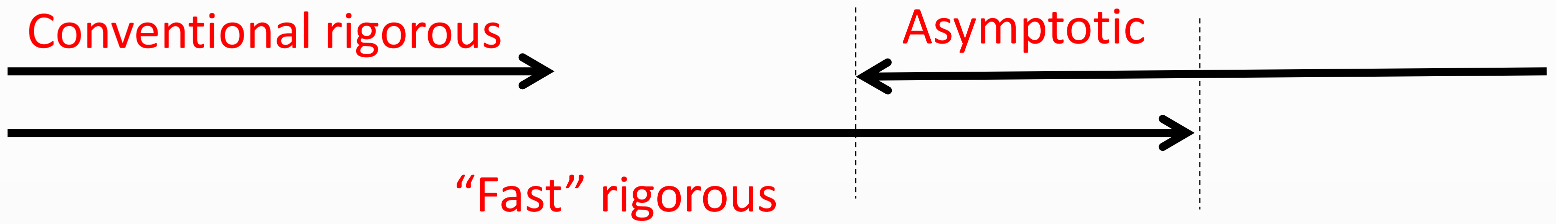
Motivation

- Rigorous numerical methods for solving problems of wave scattering, such as the Boundary Element Method (BEM), are limited to moderate frequencies and moderate scatterer sizes
- For high frequencies and large scatterers, asymptotic methods are traditionally used
- Recently developed “fast” methods, such as the MLNG approach, can extend the range of rigorous simulation to at least the lower part of the range of asymptotic methods

Motivation

- The frequency range of rigorous methods is bounded from above, and that of asymptotic methods is bounded from below



- The intersection of the ranges of “fast” rigorous and asymptotic methods opens up opportunities of their mutual validation on nontrivial examples
- In this work, an iterative MLNG-based solver for acoustic scattering problems [1] is compared with an asymptotic method [2, 3] on strongly elongated spheroids

Scalar (acoustic) scattering

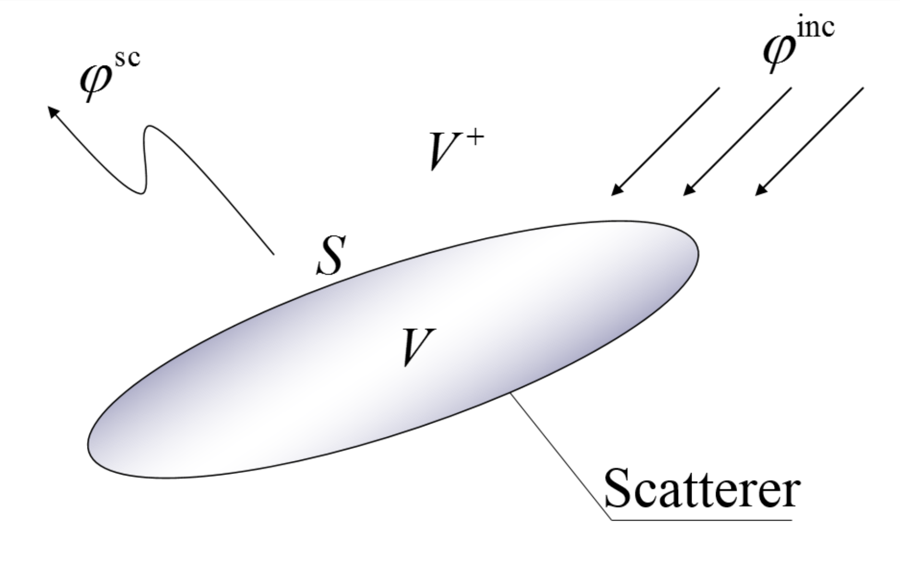
Boundary value problem

$$\varphi = \varphi^{\text{sc}} + \varphi^{\text{inc}}$$

$$\Delta\varphi(\mathbf{x}) + k^2\varphi(\mathbf{x}) = 0$$

$$\left.\frac{\partial\varphi}{\partial n}\right|_S = 0$$

$$\frac{\partial\varphi^{\text{sc}}}{\partial r} + \imath k\varphi^{\text{sc}} = o\left(\frac{1}{r}\right) \qquad \imath = \sqrt{-1}$$



Time dependence: $e^{\imath\omega t}$

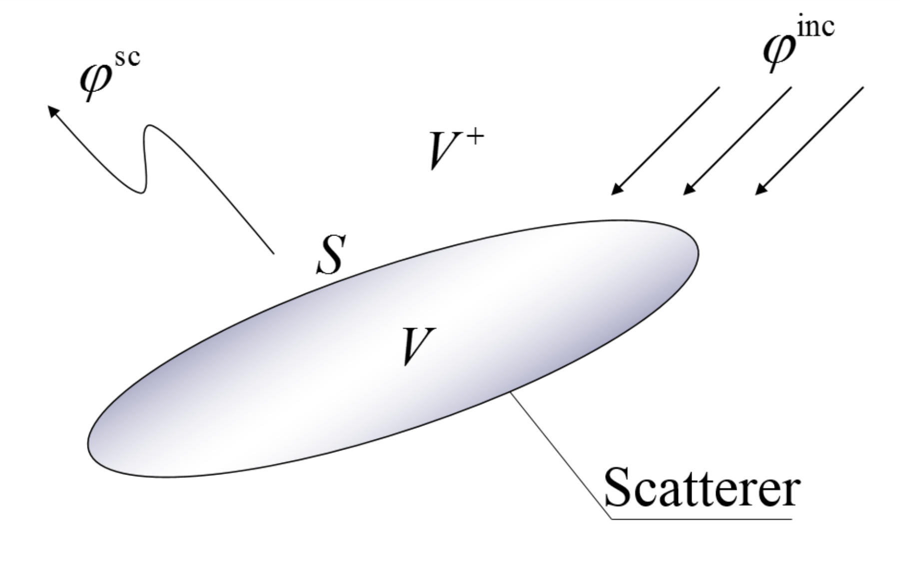
Combined-Field (Burton-Miller) Equation

$$\frac{\varphi(\mathbf{x})}{2} - \int_S \frac{\partial G(\mathbf{x}-\mathbf{x}')}{\partial n'} \varphi(\mathbf{x}') ds' + \alpha \int_S \frac{\partial G(\mathbf{x}-\mathbf{x}')}{\partial n \partial n'} \varphi(\mathbf{x}') ds' = \varphi^{\text{inc}}(\mathbf{x}) - \alpha \frac{\partial \varphi^{\text{inc}}}{\partial n}(\mathbf{x}), \quad \mathbf{x} \in S$$

$$\operatorname{Im} \alpha \neq 0$$

- uniquely solvable at all frequencies
- no spurious resonances

Stable convergence for large scatterers



Boundary Element Method

1) Triangulation of the surface
2) Piecewise-constant basis functions $\{e_n\}$

$$p = \sum_{j=1}^{N} p_n e_n$$

3) Approximation of integral operator

$$Ap = \sum_{n=1}^{N} p_n \left(Ae_n\right)$$

4) System of linear algebraic equations

$$Ap = p^{\text{inc}}$$

Computational Complexity

Conventional approach:
System of linear algebraic equations

$$\sum_{n=1}^{N} A_{mn} p_n = p_m^{\text{inc}}, \quad m = 1, \dots, N$$

N basis functions $\Rightarrow$ $N \times N$ matrix $\{A_{mn}\}$

1) Storage $O(N^2)$
2) $O(N^2)$ operations per iteration
3) Computational complexity is $O(N^2 N_{\text{iter}})$
   N_{iter} is the number of iterations

Computational Complexity

MLNG algorithm:

Fast calculation of integrals: $\Psi(\mathbf{x}) = \int_S F(\mathbf{x}, \mathbf{x}') G(\mathbf{x} - \mathbf{x}') ds'$

$G(\mathbf{x} - \mathbf{x}')$ is the Green's function; $F(\mathbf{x}, \mathbf{x}')$ is slowly oscillating

Matrix-free $\Rightarrow$

1) Storage $O(N \log N)$
2) $O(N \log N)$ operations per iteration
3) Computational complexity is $O(N \log N\ N_{\text{iter}})$

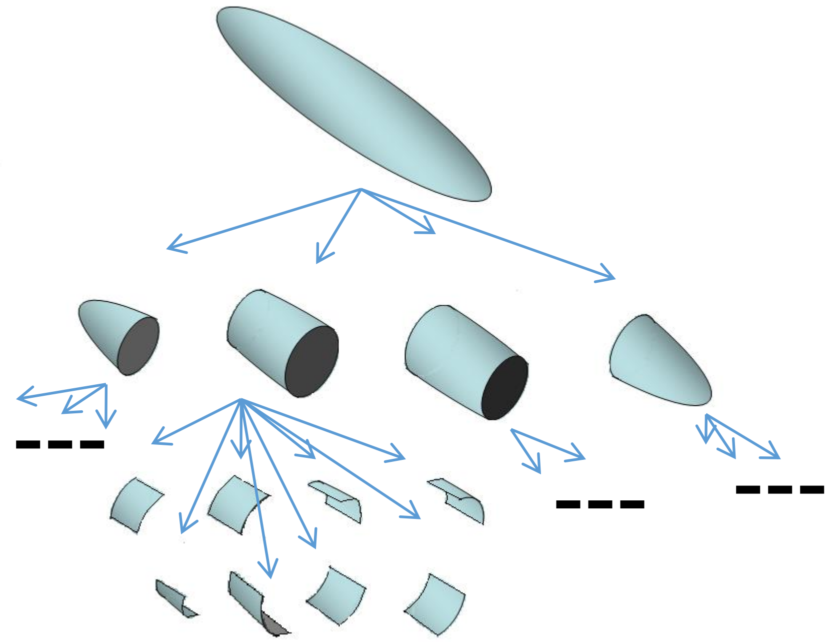
[1] E. Chernokozhin, Y. Brick, and A. Boag, "A fast and stable solver for acoustic scattering problems based on the nonuniform grid approach," J. Acoust. Soc. Am., Vol. 139, no. 1, pp. 472-480, 2016.

MultiLevel Nonuniform Grid (MLNG) algorithm

- Multilevel hierarchy of subdomains:

$$S = \bigcup_{l=2}^{L} \bigcup_{n=1}^{N_l} S_n^{(l)} \qquad \bigcap_{n=1}^{N_l} S_n^{(l)} = \varnothing \qquad S_p^{(l)} = \bigcup_{P_l(q)=p} S_q^{(l+1)}$$

- Parent-child relationships between subdomains

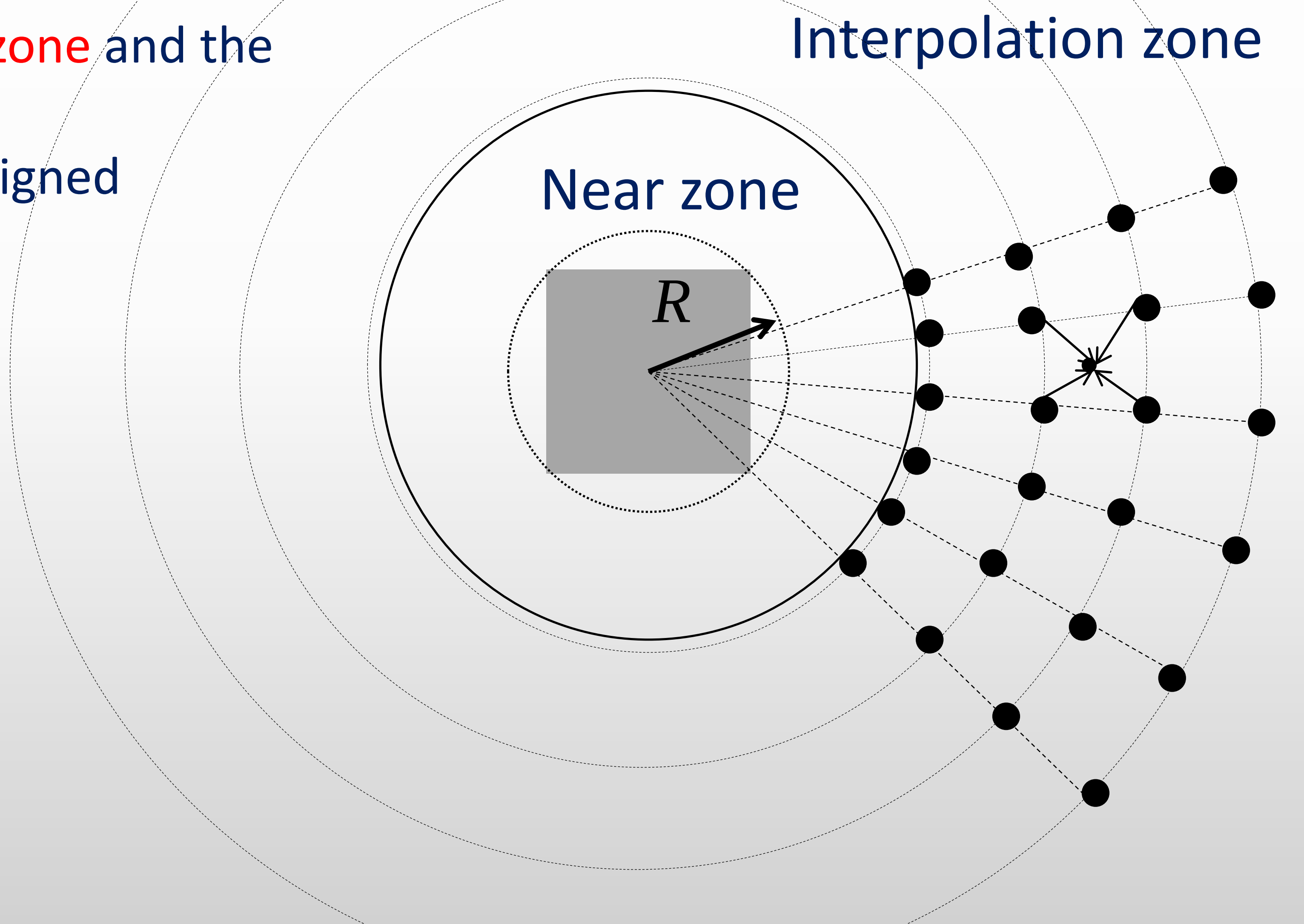


MultiLevel Nonuniform Grid Algorithm

For each partial domain, the near zone and the interpolation zone are defined and a spherical interpolation grid is assigned

Partial fields:

$$\Psi_n^{(l)}(\mathbf{x}) = \int_{S_n^{(l)}} F(\mathbf{x},\mathbf{x}')G(\mathbf{x}-\mathbf{x}')ds'$$



MultiLevel Nonuniform Grid Algorithm

Only for bottom-level domains, partial fields

$$\Psi_n^{(l)}(\mathbf{x}) = \int_{S_n^{(l)}} F(\mathbf{x},\mathbf{x}')G(\mathbf{x}-\mathbf{x}')ds'$$

are calculated directly.

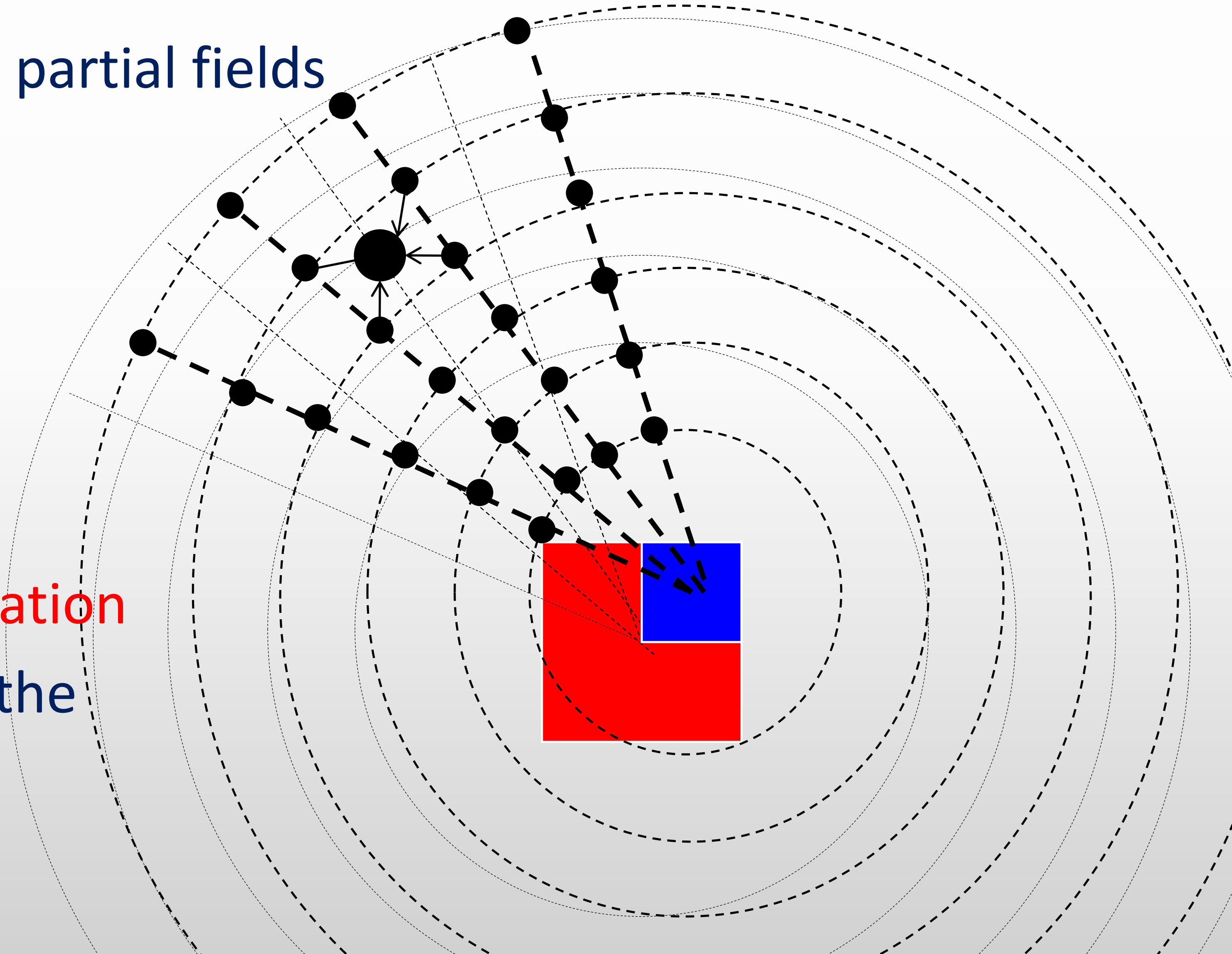
For non-bottom levels,

they are interpolated with

phase-and-amplitude compensation

from the interpolation grids of the

child subdomains



Fast calculation of integrals

$\forall \mathbf{x}, \exists O(\log N)$ nonintersecting subdomains $S_n^{(l)}(\mathbf{x})$: $S = \cup_{l,n} S_n^{(l)}(\mathbf{x})$ such that

$$\Psi(\mathbf{x}) = \int_S F(\mathbf{x},\mathbf{x}')G(\mathbf{x}-\mathbf{x}')ds' = \sum_{l,n} \int_{S_n^{(l)}(\mathbf{x})} F(\mathbf{x},\mathbf{x}')G(\mathbf{x}-\mathbf{x}')ds'$$

$\forall \int_{S_n^{(l)}(\mathbf{x})} F(\mathbf{x},\mathbf{x}')G(\mathbf{x}-\mathbf{x}')ds'$ is calculated for $O(1)$ operations:

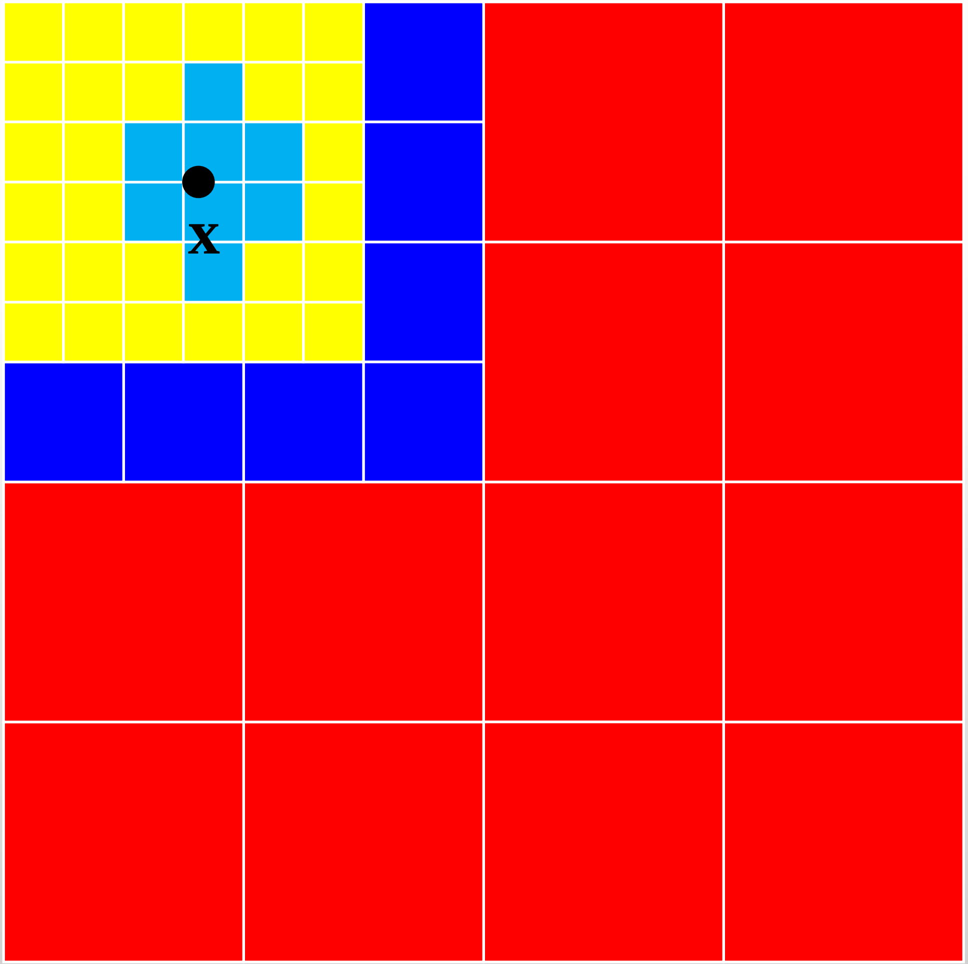
bottom level: direct integration

non-bottom levels: interpolation from sparse grids

The field at all N points,

$$\Psi = \int_S F(\mathbf{x},\mathbf{x}')G(\mathbf{x}-\mathbf{x}')ds'$$

is calculated for $O(N \log N)$ operations

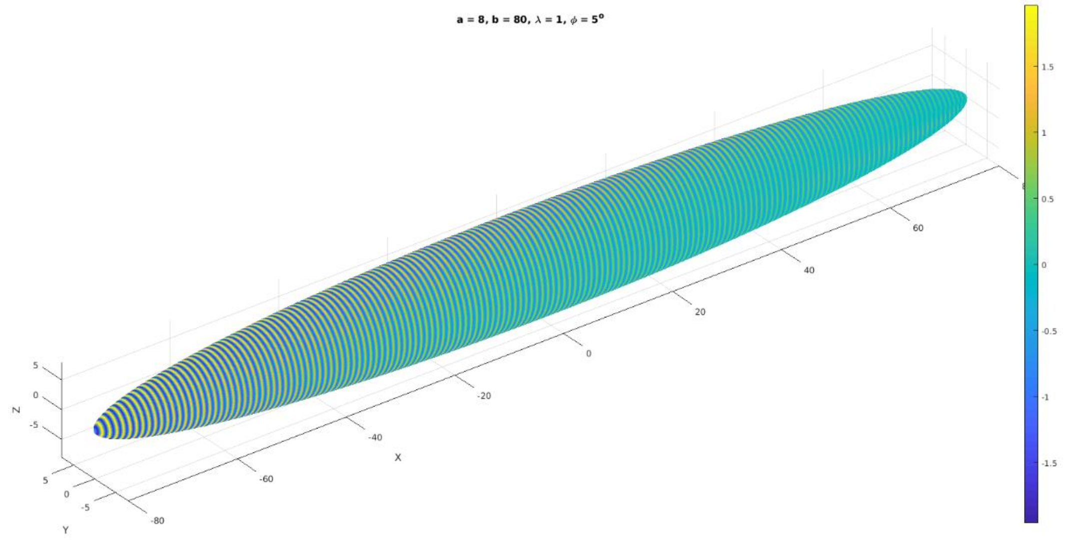


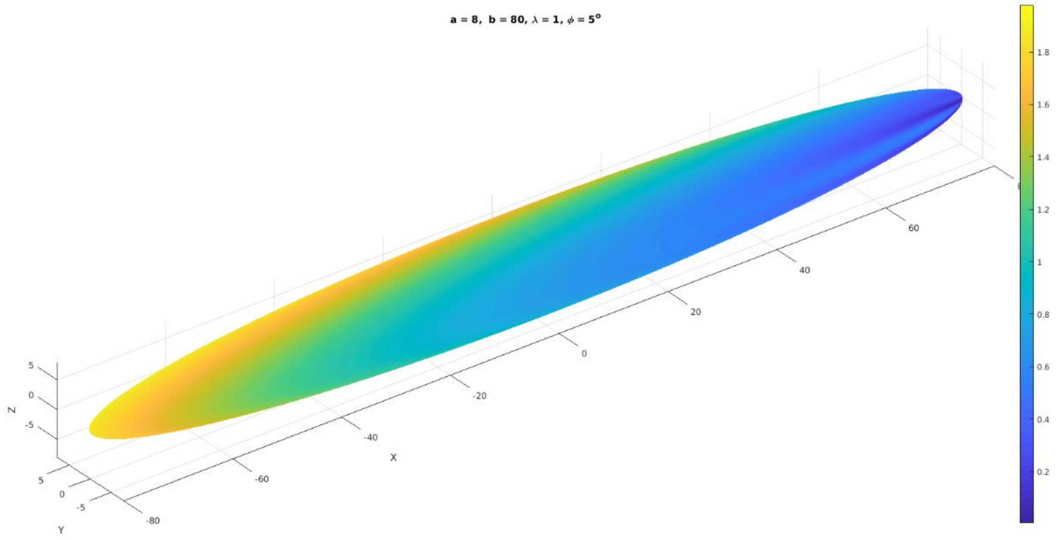
MLNG-based fast solver

MLNG algorithm + Conjugate Gradient method

- Total number of operations is $O(N \log N\, N_{\mathrm{iter}})$
- Implemented for rigid scatterers up to $N \sim 10^6$ unknowns

Example: a prolate spheroid





~1,500,000 unknowns

Asymptotic approximation

The classical V. A. Fock approximation reads

$$u = \frac{e^{iks}}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{i\sigma t} \frac{dt}{w_1^{'}(t)},$$

(k is the wave number, s is the arc-length, $\sigma = (k\rho/2)^{1/3} s/\rho$, ρ is the radius of curvature, w_1 is the Airy function in Fock notations)

However, it is not suitable in the case of too elongated bodies, because to get the desired accuracy it requires kp to be very large.

Special asymptotic approximation

Based on the assumptions:

1. High frequency, $kp>>1$ ($2p$ is the focal distance) ,
2. Strong elongation (a is minor semi-axis),

$$\chi = \frac{ka^2}{p} = O(1)$$

3. Small angle of incidence θ, namely

$$\beta = \theta\sqrt{kp} < C$$

Asymptotic procedure

1. Introduce spheroidal coordinates

$$x = p\sqrt{1-\eta^2}\sqrt{\xi^2-1}\cos\phi, \quad y = p\sqrt{1-\eta^2}\sqrt{\xi^2-1}\sin\phi, \quad z = p\eta\xi$$

2. Represent the field as the sum of the forward and backward waves, both described by parabolic equation approximation

$$u = e^{-ikp\eta}U_f(\eta,\tau,\phi) + e^{ikp\eta}U_b(\eta,\tau,\phi)$$

where τ is the scaled radial coordinate, $\tau = 2kp(\xi-1)$

The final approximation for the field on the surface of a hard spheroid:

$$u = \frac{4\sqrt{\chi}}{\pi\sqrt{1-\eta^2}} \sum_{n=0}^{+\infty} \frac{i^n \cos(n\varphi)}{n!(1+\delta_n^0)} \int_{-\infty}^{+\infty} \left(\frac{1-\eta}{1+\eta}\right)^{it} \Gamma\left(\frac{n+1}{2}+it\right) \frac{M_{it,n/2}(i\beta^2)}{\beta} \times$$

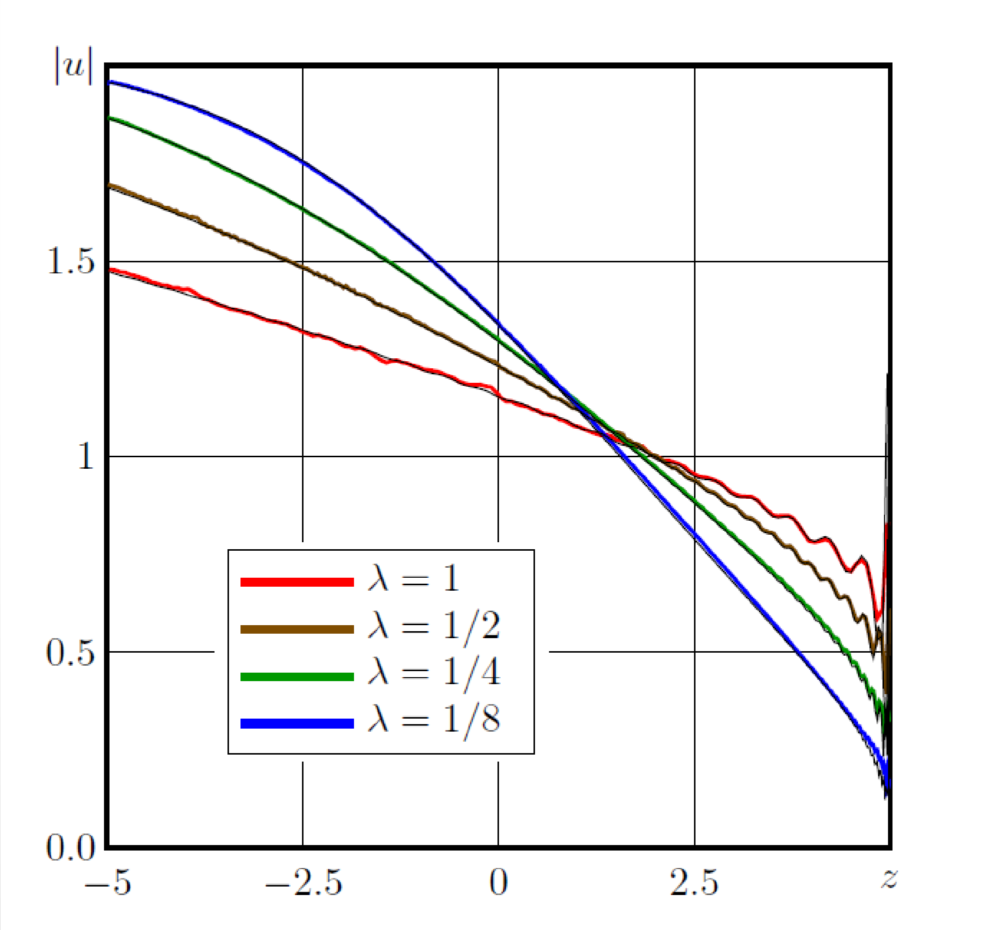
$$\times \frac{e^{-ikp\eta} + r_n(t) e^{ikp(\eta-2)}}{iW_{it,n/2}(-i\chi) - 2\chi W'_{it,n/2}(-i\chi)} dt, \quad r_n(t) = (-i)^{n+1}(4kp)^{-2it} \frac{\Gamma(\frac{n+1}{2}+it)}{\Gamma(\frac{n+1}{2}-it)}$$

Here, M and W are the Whittaker functions.

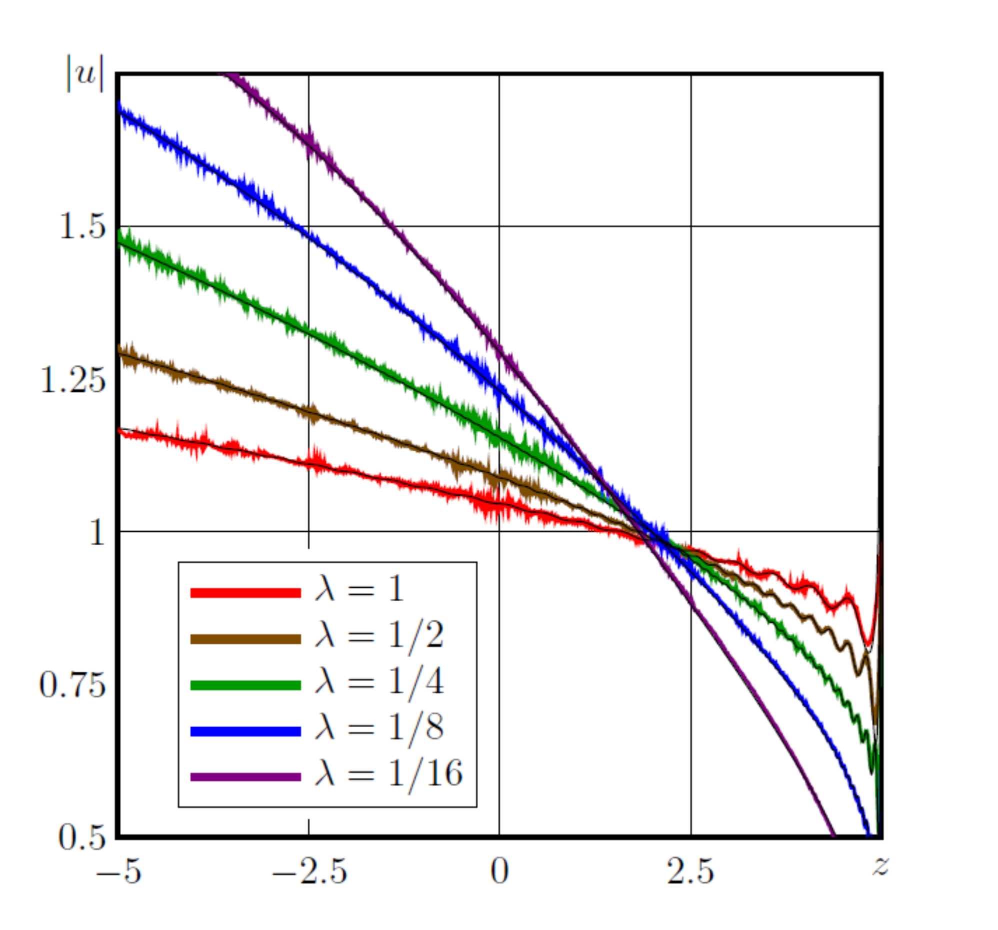
Comparison

- Comparison was performed on two hard spheroids both having length of 10 m but different aspect ratios: 1:5 (spheroid A) and 1:10 (spheroid B)
- Acoustic pressure fields on the surface excited by a plane wave at different frequencies and angles of incidence were compared

Comparison

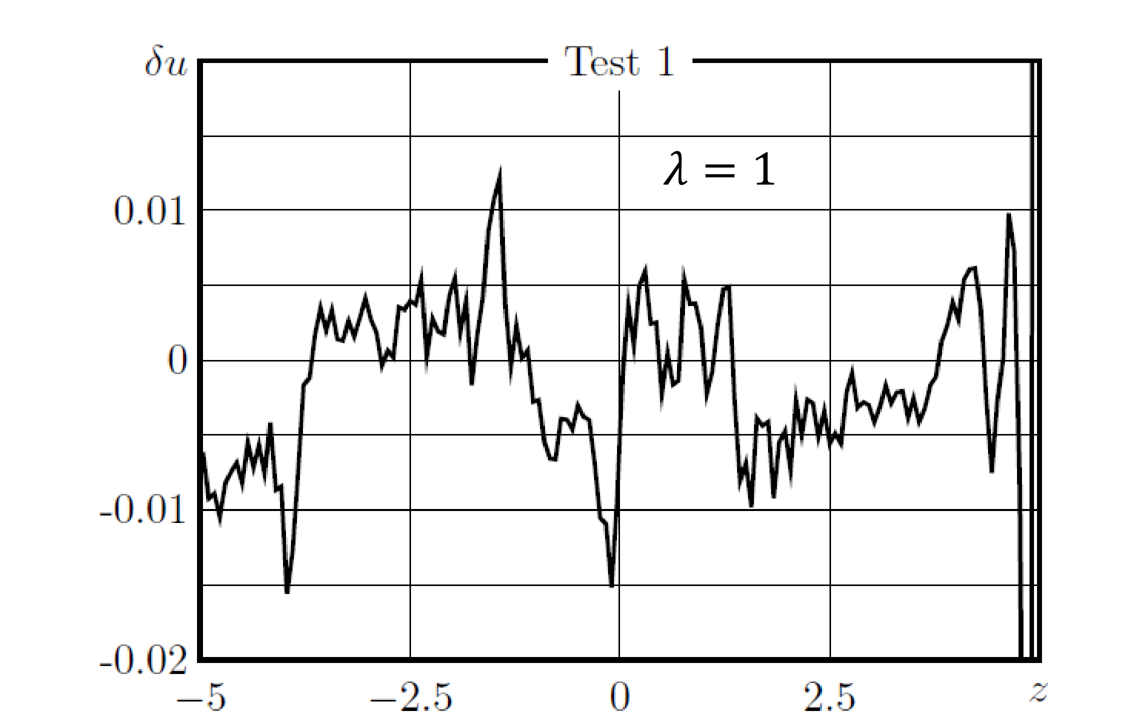


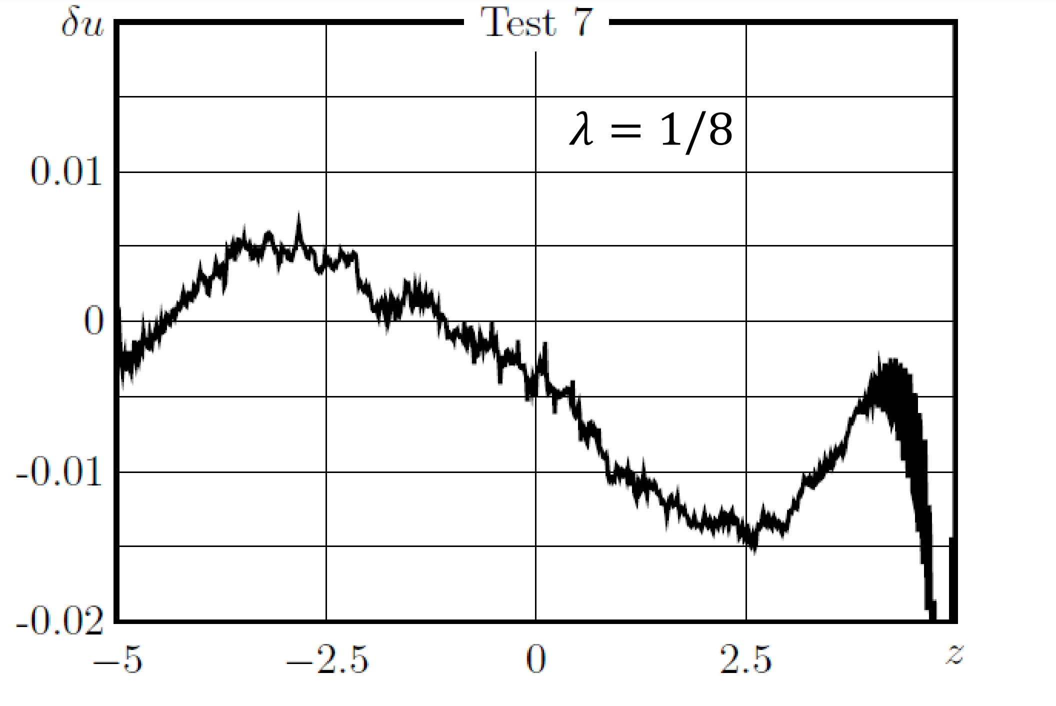
Amplitude of the field on spheroid A



Amplitude of the field on spheroid B

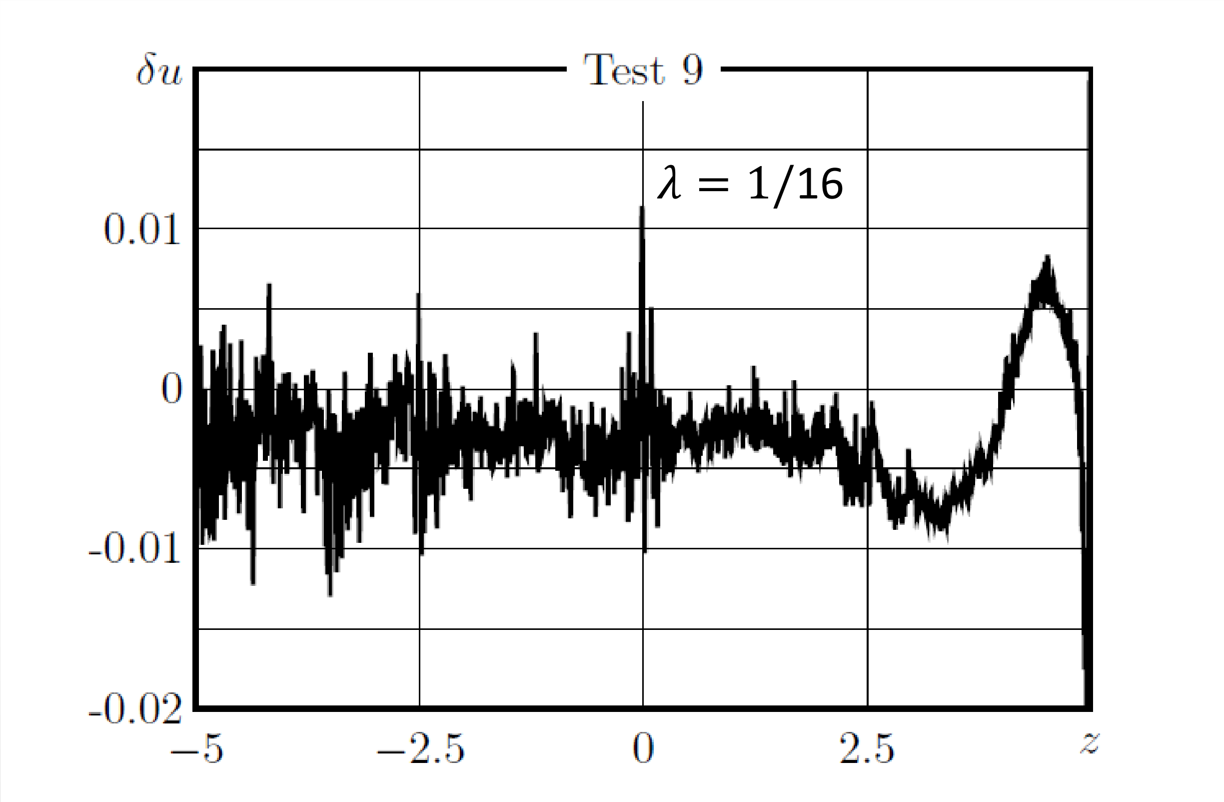
Comparison





The difference of the results for axial incidence on spheroid A

Comparison



The difference of the results for axial incidence on spheroid B

Comparison

Table 1. The ℓ_1 norms of the errors for $\lambda = 2^{(1-n)/2}$.

n	centroids	$\theta = 0^\circ$		$\theta = 10^\circ$	
Spheroid A		$\|\delta_1\|$	$\|\delta_2\|$	$\|\delta_1\|$	$\|\delta_2\|$
1	11232	0.0072	0.0192	0.0077	0.0207
2	21624	0.0078	0.0217	0.0082	0.0235
3	42076	0.0082	0.0231	0.0083	0.0255
4	85816	0.0081	0.0268	0.0082	0.0299
5	169344	0.0085	0.0283	0.0087	0.0329
6	328852	0.0075	0.0433	0.0084	0.0361
7	644252	0.0087	0.0419	0.0085	0.0420
8	1261324	0.0088	0.0720	0.0079	0.0327

n	centroids	$\theta = 0^\circ$		$\theta = 10^\circ$	
Spheroid B		$\|\delta_1\|$	$\|\delta_2\|$	$\|\delta_1\|$	$\|\delta_2\|$
1	10486	0.0019	0.0049	0.0028	0.0064
2	10486	0.0036	0.0074	0.0036	0.0076
3	44180	0.0017	0.0040	0.0021	0.0063
4	44180	0.0032	0.0060	0.0029	0.0063
5	88124	0.0038	0.0074	0.0038	0.0123
6	174530	0.0039	0.0077	0.0037	0.0085
7	355243	0.0041	0.0088	0.0051	0.0204
8	714224	0.0039	0.0090	0.0044	0.0106
9	1476190	0.0048	0.0111	0.0059	0.0362

Conclusions

- The comparison has shown that the difference between the surface fields calculated using the MLNG-based solver and the asymptotic approximation is generally within 1% in the integral norm if only the absolute values are compared and about twice as large if the phases are also taken into account

- Most of the deviation, which proved to be about 1%, is due to the numerical noise caused by the use of the zeroth order basis functions in the MLNG-based algorithm

- This fairly good agreement demonstrates the accuracy of both methods and, in particular, provides a nontrivial validation for the MLNG-based solver, which in turn supports its application to a much wider class of scatterers

References

[1] E. Chernokozhin, Y. Brick, and A. Boag, “A fast and stable solver for acoustic scattering problems based on the nonuniform grid approach,” J. Acoust. Soc. Am. **139** (1), 472-480 (2016).

[2] I. V. Andronov, “Diffraction by a strongly elongated body of revolution,” Acoust. Phys. **57** (2), 121-126 (2011).

[3] I. V. Andronov, “High-frequency acoustic scattering from prolate spheroids with high aspect ratio,” J. Acoust. Soc. Am. **134** (6), 4307-4316 (2013).

[4] I. V. Andronov, D. Bouche, and M. Duruflé, “High-frequency currents on strongly elongated spheroids,” IEEE Trans. Antennas Propag. **65** (2), 794-804 (2017).