# An asymptotic evaluation of a kernel in the study of a radiation operator: the strip current in near zone

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## Abstract

In this paper we consider a radiation operator $T$ in near non-reactive zone, and we study the analytical properties of the operator $TT^\dagger$. In particular, first we perform an asymptotic evaluation of the kernel of $TT^\dagger$; later, we derive a closed-form approximation of the eigenfunctions of such a operator which is valid on a subset of the observation domain. The study is done with reference to a magnetic strip current whose radiated electric field is observed on a bounded observation domain parallel to the source.

## 1 Introduction

The singular values decomposition of an operator represents a key mathematical tool in linear inverse problems [1]. The relevance of the singular system is due essentially to the following reasons. First of all, it allows expressing the unknown function of the linear problem explicitly. In the second place, the singular system is related to some figures of merit which are usually exploited in the evaluation of the performance that can be achieved in linear inverse problems. Such figures of merit involve the number of degrees of freedom (*NDF*), and the point spread function (*PSF*).
Unfortunately, in some cases the singular system of the relevant operator is not known in closed-form; in such cases, numerical computation of the singular system must be performed. Despite this, the numerical computation of the eigenspectrum does not allow to show the role played by the geometry, and by the configuration parameters; for this reason, a closed-form expression of the singular system is always desirable.
In this paper, we consider the inverse source problem correspondent to a magnetic strip current whose electric field is observed on a bounded domain parallel to the source located in near non-reactive zone. For such near zone configuration, the study of the singular system of the radiation operator has been recently addressed in [3] [4]. However, in the above mentioned papers only the right singular functions of the radiation operator are computed in a closed form, instead, the left singular functions are computed by performing the image of the right singular function through the radiation operator.
Here, we follow a different methodology for the study of the radiation operator which is based on an asymptotic approach. In particular, we first evaluate the kernel of the eigenvalue problem for the computation of the left singular values; and later, we provide a closed-form approximation of the left singular functions of the radiation operator which works on the portion of the observation domain in front of the source.

## 2 Geometry of the problem

Consider the $2D$ geometry depicted in fig. 1 where a magnetic current $\underline{J}_m$, directed along the $y$ axis and supported on the set $[-a,a]$ of $x$ axis, radiates in a homogeneous medium with wavenumber $\beta$. Suppose to observe the $x$ component of the electric field radiated by such current on a bounded observation domain $OD = [-X_o, X_o]$ that is parallel to the source and located at a distance $z_0$.

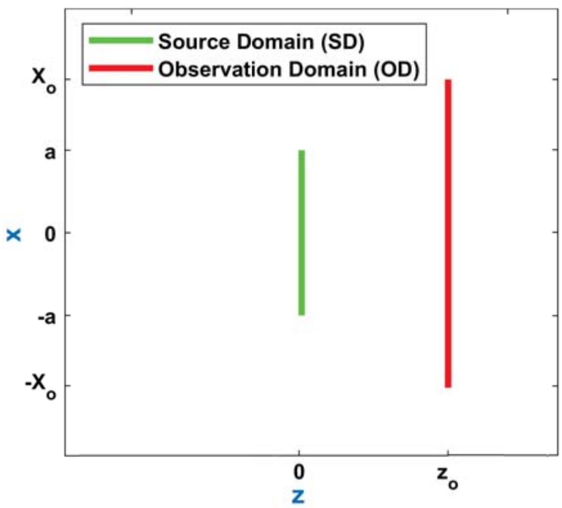


**Figure 1.** Geometry of the problem

For the configuration at hand, the $x$ component of the electric field has the following mathematical expression

$$E(x_o) = \int_{-a}^{a} g(x_o,x')\, J_m(x')\, dx' \tag{1}$$

where the Green function $g(x_o,x')$ in near non-reactive zone can be approximated (apart for a constant factor) as

$$g(x_o,x') \approx \frac{z_o}{R^{\frac{3}{2}}(x_o,x')}\, e^{-j\beta R(x_o,x')} \tag{2}$$

with $R(x_o,x') = \sqrt{(x_o-x')^2+z_o^2}$.

Equation (1) provides an explicit form of the radiation operator $T$ which is such that

$$T : J_m \in L^2[-a,a] \longrightarrow E \in L^2[-X_o,X_o] \tag{3}$$

with $L^2[-a,a]$, and $L^2[-X_o,X_o]$ indicating the sets of square integrable functions on which the radiation operator acts. Consequently, the adjoint operator $T^\dagger$ is defined as

$$T^\dagger : E \in L^2[-X_o,X_o] \longrightarrow J_m \in L^2[-a,a] \tag{4}$$

and it can be expressed by $T^\dagger E = \int_{-X_0}^{X_0} g^*(x_o,x')\,E(x_o)\,dx_o$.

## 3 The kernel properties of $TT^\dagger$

In this section we study the kernel properties of the operator $TT^\dagger$. After that, we provide an approximate expression of the left singular functions of $T$ which works well in the subset of the observation domain located in front of the source. In order to do as said above, let us write the operator $TT^\dagger$ in following explicit form

$$TT^\dagger E = \int_{-X_0}^{X_0} K(x,x_o)E(x_o)\,dx_o \tag{5}$$

where the kernel $K(x,x_o)$ is given by

$$K(x,x_0) = \int_{-a}^{a} g(x,x')g^*(x_o,x')\,dx' \tag{6}$$

In order to obtain a closed-form expression of the kernel, let us substitute eq. (2) in (6). By doing this, we have that

$$K(x,x_o) = z_o^2 \int_{-a}^{a} \frac{e^{-j\beta\,[R(x,x')-R(x_o,x')]}}{R^{\frac{3}{2}}(x,x')R^{\frac{3}{2}}(x_o,x')}\,dx' \tag{7}$$

By setting $f(x',x,x_o) = 1/[R^{\frac{3}{2}}(x,x')R^{\frac{3}{2}}(x_o,x')]$ and $\phi(x',x,x_o) = [R(x,x') - R(x_o,x')]/a$, the kernel $K(x,x_o)$ can be expressed by the following integral

$$K\,(x,x_o) = z_o^2 \int_{-a}^{a} f(x',x,x_o)\,e^{-j\beta\,a\,\phi(x',x,x_o)}\,dx' \tag{8}$$

The integral above can be evaluated by exploiting the integration by parts method. The latter provides the following relation

$$K = \frac{-z_o^2}{j\beta a}\left(\frac{f}{\phi'}\,e^{-j\beta\,a\,\phi}\right)_{-a}^{a} + \frac{z_o^2}{j\beta a}\int_{-a}^{a}\frac{d}{dx'}\left(\frac{f}{\phi'}\right)e^{-j\beta\,a\,\phi}\,dx' \tag{9}$$

For $\beta a >> 1$ the integral in (9) goes to zero for the Riemann Lebesgue lemma; consequently, the second term in (9) is an $o\,(1/\beta)$ while the first term is an $O(1/\beta)$ [2]. This imply that for $\beta\,a >> 1$ the second term in (9) can be neglected, and $K\,(x,x_o)$ can be approximated as

$$K\,(x,x_o) \approx -\frac{z_o^2}{j\beta a}\left(\frac{f_a}{\phi_a'}\,e^{-j\,\beta\,a\,\phi_a} - \frac{f_{-a}}{\phi_{-a}'}\,e^{-j\,\beta\,a\,\phi_{-a}}\right) \tag{10}$$

where the subscripts $a$ or $-a$ denote that the correspondent function has been computed in the point $x' = a$ or $x' = a$. The result expressed by (10) holds $\forall x \neq x_o$. The actual value of $K$ in $x = x_o$ can be obtained by particularizing the integral (7) for $x = x_o$, and by evaluating it. From this, it follows that

$$K(x_o,x_o) = \frac{x_o + a}{R(x_o,-a)} - \frac{x_o - a}{R(x_o,a)} \tag{11}$$

Anyway, the asymptotic evaluation (10) connects in a continuous way with the actual value of the kernel in $x = x_o$. Nevertheless equation (10) represents an asymptotic evaluation of the actual kernel $K(x,x_o)$ that works very well in practice, the properties of a linear integral operator with such expression of the kernel are not available in literature. With the aim to approximate $K(x,x_o)$ by a kernel that has been already studied in literature, let us approximate the amplitude terms $(f_a/\phi_a')$ and $(f_{-a}/\phi_{-a}')$. Since the point $x = x_o$ is a pole of order 1 for the functions $(f_a/\phi_a')$ and $(f_{-a}/\phi_{-a}')$, such functions can be represented through a Laurent series. By truncating the series, we obtain that

$$\frac{f_a(x,x_o)}{\phi_a'(x,x_o)} \approx \frac{f_{-a}(x,x_o)}{\phi_{-a}'(x,x_o)} \approx -\frac{a}{z_o^2(x-x_o)} \tag{12}$$

For an observation domain in near zone, approximation (12) works well in almost all the points $(x,x_o) \in [-a,a] \times [-a,a]$. Instead, when the distance $z_o$ increases, the approximation holds in a more extended region. Figure 2 shows the reasonableness of the approximation above.

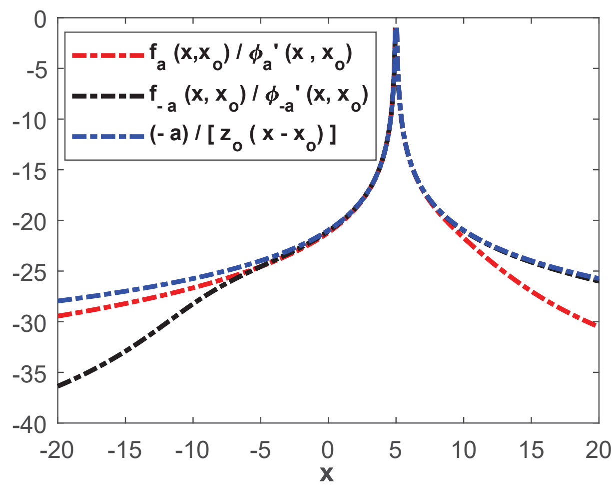


**Figure 2.** Behavior of $\left|\frac{f_a(x,x_o)}{\phi_a'(x,x_o)}\right|$, $\left|\frac{f_{-a}(x,x_o)}{\phi_{-a}'(x,x_o)}\right|$, $\left|-\frac{a}{z_o^2(x-x_o)}\right|$ in dB for $a = 10\lambda$, $x_o = 5\lambda$, $z_o = 5\lambda$.

By exploiting eq. (12), $K\,(x,x_o)$ can be written in the form

$$K\,(x,x_o) \approx \frac{e^{-j\,\frac{\beta a}{2}\left(\phi_{-a}+\phi_a\right)}}{\beta\,(x-x_o)}\,sin\left(\frac{\beta a}{2}\left(\phi_{-a} - \phi_a\right)\right) \tag{13}$$

The last expression of $K\,(x,x_o)$ suggests to introduce the following transformation

$$\eta_o = \eta\,(x_o) = \frac{1}{2a}\left(\sqrt{(x_o+a)^2 + z_o^2} - \sqrt{(x_o-a)^2 + z_o^2}\right) \tag{14}$$

Such transformation is differentiable, invertible, and it makes the kernel more similar to a convolution one. In fact, transformation (14) allows expressing the operator $TT^{\dagger}$ in the form

$$TT^{\dagger}E = \int_{\eta(-X_0)}^{\eta(X_0)} K(\eta,\eta_o)\,E(\eta_o)\,d\eta_o \tag{15}$$

where

$$K(\eta,\eta_o) = \frac{dx}{d\eta_o}\,\frac{e^{-j\frac{\beta a}{2}\left(\phi_{-a}(\eta,\eta_o)+\phi_a(\eta,\eta_o)\right)}}{\beta\,(x(\eta)-x(\eta_o))}\,sin\big(\beta a\,(\eta-\eta_o)\big) \tag{16}$$

Note that $\eta_o$ represents one of the two elliptical coordinates. By introducing the other elliptical coordinate $\gamma_o$, which is defined as

$$\gamma_o = \gamma(x_o) = \frac{1}{2a}\left(\sqrt{(x_o+a)^2+z_o^2}+\sqrt{(x_o-a)^2+z_o^2}\right), \tag{17}$$

it results that $\frac{\beta a}{2}\left(\phi_{-a}(\eta,\eta_o)+\phi_a(\eta,\eta_o)\right) = \beta a\left(\gamma(\eta)-\gamma(\eta_o)\right)$. Hence, the kernel $K(\eta,\eta_o)$ can be rewritten as

$$K(\eta,\eta_o) \approx \frac{\pi}{\beta}A(\eta,\eta_o)\,e^{-j\beta a\left(\gamma(\eta)-\gamma(\eta_o)\right)}\,\frac{sin\big(\beta a\,(\eta-\eta_o)\big)}{\pi(\eta-\eta_o)} \tag{18}$$

where the amplitude function $A(\eta,\eta_o)$ is given by

$$A(\eta,\eta_o) = \frac{dx}{d\eta_o}\,\frac{\eta-\eta_o}{x(\eta)-x(\eta_o)} \tag{19}$$

At this point note that apart for the amplitude function $A(\eta,\eta_o)$, the kernel is very similar to a convolution kernel of sinc type which is well known in literature. An exactly sinc expression of the kernel can be obtained in the region where the following approximation works

$$\frac{\eta-\eta_o}{x(\eta)-x(\eta_o)} \approx \frac{1}{\frac{dx}{d\eta}\big|_{\eta=\eta_o}} \tag{20}$$

Naturally, the region of validity of (20) depends on the behavior of the function $x(\eta)$. In particular, approximation (20) works well if $x(\eta)$ is well approximated by a linear function around the point $\eta=\eta_o$. By computing the inverse of $\eta(x) = \frac{1}{2a}\left(\sqrt{(x_o+a)^2+z_o^2}-\sqrt{(x_o-a)^2+z_o^2}\right)$, we obtain that $x(\eta)$ is given by

$$x(\eta) = \eta\sqrt{a^2+\frac{z_o^2}{1-\eta^2}} \tag{21}$$

Fig. 3 sketches $x(\eta)$ for different values of $z_o$. As can be seen from such figure, $x(\eta)$ exhibits a linear behavior for little value of $|\eta|$, and then it bends at the ends.

Furthermore, for little values of $z_o$ (i.e, for an observation domain in near zone), $x(\eta)$ is well approximated by a linear function only for the values of $\eta$ mapped into the values of $x$ belonging to the set $[-a,a]$. Instead, if $z_o$ increases, approximation (20) works well in a more extended region.

From as said above, it follows that for an observation domain in near zone $A(\eta,\eta_o)\approx 1$ only for $(\eta,\eta_o)\in[\eta(-a),\eta(a)]\times[\eta(-a),\eta(a)]$. Consequently, in such region, the kernel $K(\eta,\eta_o)$ can be approximated by a convolution kernel whose expression is given by

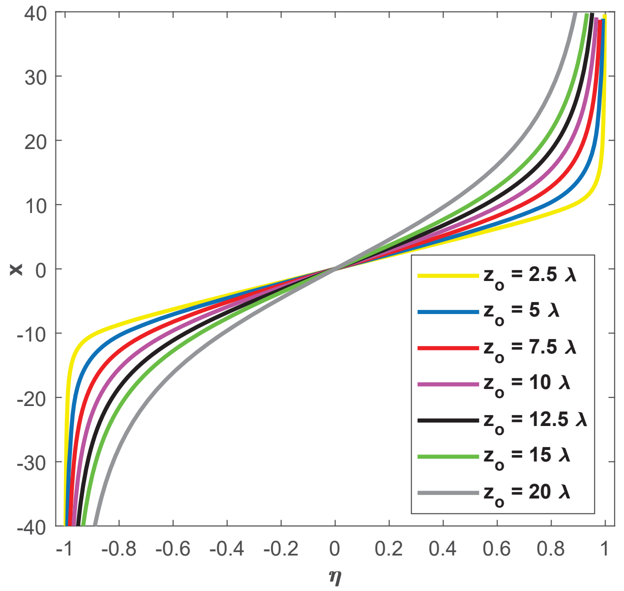


**Figure 3.** Diagram of $x(\eta)$ for different values of the distance $z_o$ in the case of a source whose semi-extension $a=10\lambda$.

$$K(\eta,\eta_o) \approx \frac{\pi}{\beta}\,e^{-j\beta a\left(\gamma(\eta)-\gamma(\eta_o)\right)}\,\frac{sin\big(\beta a\,(\eta-\eta_o)\big)}{\pi(\eta-\eta_o)} \tag{22}$$

Figure 4 shows the actual kernel of the operator $TT^{\dagger}$ (numerically computed), the asymptotic evaluation, and the sinc approximation in the domain $(\eta,\eta_o)$. As appear from the figure, the actual kernel and the asymptotic evaluation have the same behavior $\forall(\eta,\eta_o)$, instead, the the sinc approximation has the same behavior of the actual kernel only in most part of the domain $[\eta(-a),\eta(a)]\times[\eta(-a),\eta(a)]$. The latter is highlighted in fig. 4 by the square in red.

Figure 5 exhibits the actual behavior, the asymptotic evaluation, and the sinc approximation of $K(\eta)$ for two different values of $\eta_o$, the first inside the set $[\eta(-a),\eta(a)]$ and the second outside. Note that if $\eta_o\notin[\eta(-a),\eta(a)]$, since the approximations made above are not true, the sinc approximation does not work.

The expression of the kernel $K(\eta,\eta_o)$ provided by (22) allows to find a closed-form approximation of the left singular functions $\{v_n(\eta)\}$ of the radiation operator $T$. In fact, since the kernel of the eigenvalue problem $TT^{\dagger}v_n(\eta_o)=\sigma_n^2 v_n$ is the same of a Slepian Pollak operator, it results that for $\eta\in[\eta(-a);\eta(a)]$ the left singular functions $v_n(\eta)$ can be approximated by

$$v_n(\eta) \approx \frac{\psi_n(\eta,c)}{\sqrt{\lambda_n(c)}}\,e^{j\beta a\gamma(\eta)}$$

where $c=\beta a\,\eta(X_o)$ is the so-called spatial bandwidth product; $\psi_n(\eta,c)$ are the prolate spheroidal waves functions (PSWFs); $\lambda_n=\beta\sigma_n^2/\pi$ are the eigenvalues of the PSFWs.

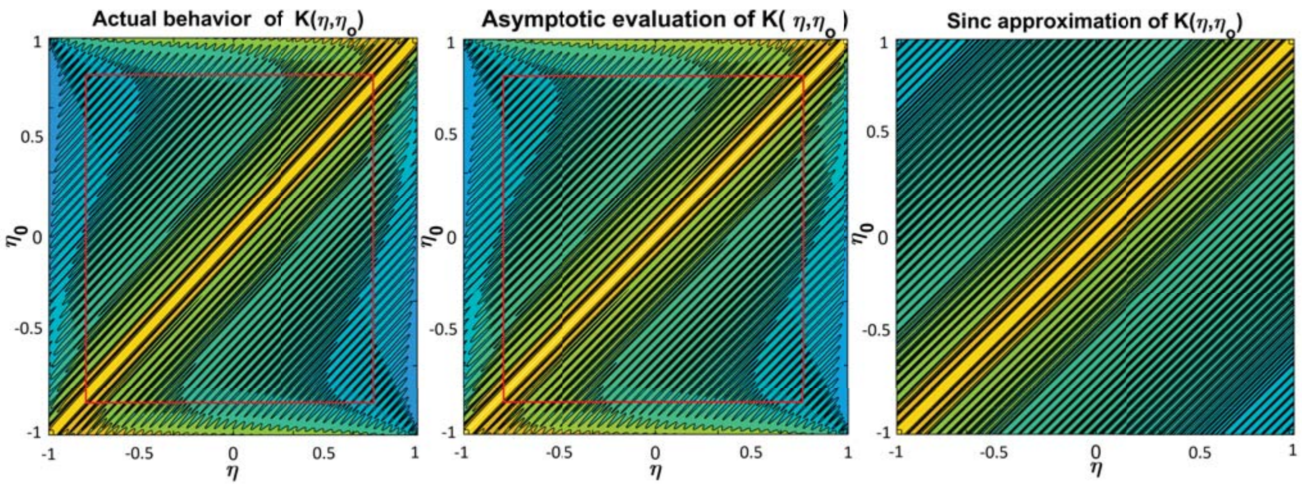


**Figure 4.** Actual kernel of the operator $TT^\dagger$, asymptotic evaluation, and sinc approximation for $a = 10\lambda$ $\left(\eta(a) = 0.781\right)$, $z_o = 5\lambda$.

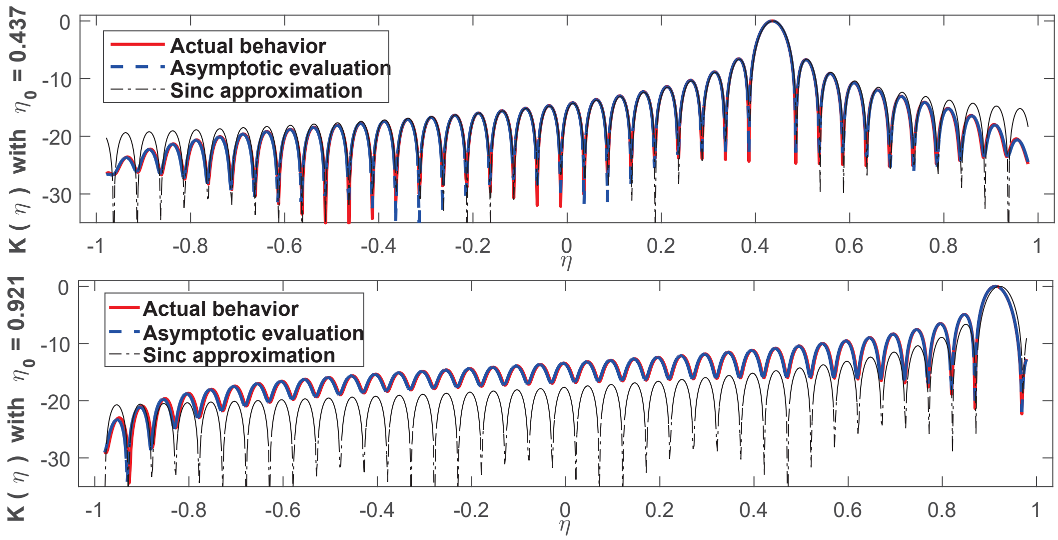


**Figure 5.** Actual behavior, asymptotic evaluation, and sinc approximation of $K(\eta)$ for $\eta_o = 0.437$ ($x_o = 5\lambda$) and $\eta_o = 0.921$ ($x_o = 15\lambda$) in the case in which $a = 10\lambda$ $\left(\eta(a) = 0.781\right)$ and $z_o = 5\lambda$.

## 4 Conclusions

In this paper a $2D$ geometry consisting of a magnetic strip current observed on a bounded domain parallel to the source has been considered. With reference to such configuration, the operator $TT^\dagger$ (which is involved in the eigenvalue problem for the computation of the left singular functions of the radiation operator $T$) has been studied. In particular, first the kernel of the operator $TT^\dagger$ has been evaluated by exploiting an asymptotic approach. Later, by performing some approximations and a suitable change of variables, the kernel of the operator $TT^\dagger$ has been recast as convolution kernel of sinc type. This has allowed to give a closed-form approximation of the left singular functions $v_n$ of the radiation operator $T$ which works on the subset of the observation domain located in front of the source.

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