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# First- and Second Order Characterization of Temporal Moments of Stochastic Multipath Channels

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## Abstract

The covariance structure of temporal moments for stochastic multipath channel models are derived in closed from. The obtained expression shows the importance of both the arrival rate and the second factorial intensity function of the point process describing the multipath model.

## 1 Introduction

Stochastic multipath models are widely used in the communication community to analyse and simulate various system aspects considering wideband channels. The delay dispersion due to a wideband channel is commonly characterised by use of the rms delay spread and thus distributions of rms delay spread are commonly reported in measurement and simulation studies. [1]. The rms delay spread is defined based on the first three raw temporal moments.

For stochastic multipath models, the temporal moments are random variables with mean value and covariances. The means of temporal moments, are given by the power delay spectrum and are simple to derive [2]. However, general expressions for the covariance of temporal moments have to the author's knowledge not been published.

Recently, a multivariate log-normal for the temporal moments was found to represent two sets of measurement data well [3]. The log-normal model is easy to fit to multivariate data, as it is fully specified by the mean and covariances. Thus to apply such a log-normal model the mean and covariances of the temporal moments should be known. Temporal moments have recently been applied in calibration of stochastic multipath models [4, 5] eliminating the need for multipath extraction. For this application, statistical characterization is relevant for deriving parameter estimators and evaluating their accuracy.

In the present contribution, we derive a general expression for the covariance of arbitrary temporal moments for stochastic multipath models. The general expression is then applied to the model by Turin et al. [6]. Turin's model is has the taken as building block for numerous stochastic channel models including those by Saleh-Valenzula, Spencer and several recent works including [7, 8, 2]. Despite its importance and simplicity, the literature holds surprisingly few results on properties of Turin's model. In particular, we have not been able to find published result on the covariances of temporal moments for this model. Therefore, we derive these in the following first for general setting and then for specific room electromagnetic setting of the model.

## 2 Stochastic Multipath Models

The instantaneous (unnormalized and uncentered) temporal moment of order $i$ is defined as

$$m_i = \int_0^\infty |y(\tau)|^2 \tau^i d\tau, \tag{1}$$

where $y(t)$ denotes the complex baseband representation of the received signal. In a multipath model, the received signal is a superposition of signal components thought of as arriving via multiple distinct paths.

$$y(t) = \sum_{x \in \mathscr{X}} \alpha_x s(t - \tau_x). \tag{2}$$

where $x$ denotes the pair $(\tau_x, \alpha_x)$ of delay $\tau_x$ and complex gain $\alpha_x$. The countable, but possibly infinite, collection of pairs $\mathscr{X} = \{x_1, x_2, x_3, \dots\}$ is a marked point process with (unordered) points $\{\tau_x\}$ and associated marks $\{\alpha_x\}$. The intensity function, or arrival rate, is denoted by $\lambda(t)$. The above formulation is not limited to models that were formulated as marked point processes, for example the models [6] and [9], but is more widely applicable to multipath models [10, 11, 2]. The point process formulation simplifies derivation by use of theorems from the rich literature on stochastic geometry [12, 13, 14].

Assuming the complex gains to be uncorrelated random variables, i.e. the uncorrelated scattering (US) assumption, the second moment of the received signal can be written in terms of the delay power spectrum $P(t)$ as

$$\mathbb{E}[|y(\tau)|^2] = \int_{-\infty}^\infty P(\tau - t)|s(t)|^2 dt. \tag{3}$$

The delay power spectrum can be obtained as a product of the arrival rate $\lambda(\tau)$ and the conditional second moment of the complex gains $\sigma_\alpha^2(\tau) = \mathbb{E}[|\alpha|^2|\tau]$, i.e.

$$P(\tau) = \sigma_\alpha^2(\tau)\lambda(\tau). \tag{4}$$

For causal models, $\lambda(\tau) = 0$ and thus $P(\tau) = 0$ for $\tau < 0$.

The temporal moments simplify if the duration of the transmitted signal is short, i.e. $|s(t)|^2$ approaches a Dirac delta impulse. Under this high-bandwidth assumption,

$$m_i = \sum_{k=0}^{\infty} |\alpha_k|^2 \tau_k^i, \quad i = 0, 1, 2, \dots \tag{5}$$

The temporal moments of a stochastic model are joint random variables with mean and covariances.

## 3 Mean and Covariance of Temporal Moments

The mean and covariance of the temporal moments, can be computed For the derivation we draw upon tools from the theory of point processes, in particular the Campbell theorem. The reader is referred to [12, 13, 14] for details on the point processes.

Under the US assumption, Campbell's theorem gives the wellknown result for the mean,

$$\mu_i = \mathbb{E}[m_i] = \int \sigma_\alpha^2 \lambda(t) t^i dt = \int P(t) t^i dt, \quad i = 0, 1, 2, \dots \tag{6}$$

Thus the power delay spectrum specifies the means of all the temporal moments.

To derive the covariance structure we start from

$$\sigma_{ij} = \mathbb{E}[m_i m_j] - \mu_i \mu_j. \tag{7}$$

Applying the law of total expectation leads to

$$\mathbb{E}[m_i m_j] = \mathbb{E}\left[\sum_{x,x'} A(\tau_x, \tau_{x'}) \tau_x^i \cdot \tau_{x'}^j\right] \tag{8}$$

where the fourth moments of the complex gains enter in the function $A(\tau_x, \tau_{x'}) = \mathbb{E}[|\alpha_x|^2 |\alpha_{x'}|^2 | \tau_x, \tau_{x'}]$. The double sum in (8) contains "diagonal" and "cross" terms. The expectation of the diagonal terms follows by Campbell's theorem

$$\mathbb{E}\left[\sum_x A(\tau_x, \tau_x) \tau_x^{i+j}\right] = \int A(\tau, \tau) \tau^{i+j} d\tau. \tag{9}$$

The expectation of the sum of cross terms can be carried out using the second order Campbell formula as

$$\mathbb{E}\left[\sum_{x \neq x'} A(\tau_x, \tau_{x'}) \tau_x^i \cdot \tau_{x'}^j\right] = \iint A(\tau, \tau') \lambda^{(2)}(\tau, \tau') \tau^i \tau'^j d\tau d\tau'. \tag{10}$$

where $\lambda^{(2)}(\tau, \tau')$ is the so-called second order factorial intensity function of the arrival time point process.

The above result can be used to compute the covariance of the temporal moments for any model where the fourth moment structure of the gains and the second order factorial intensity are known. The following example shows the derivation for the specific case of an Turin's model.

## 4 Application to Turin's Model

In the most general setting, Turin's model [6] can be formulated as an independently marked Poisson process $\mathscr{X}$. The arrival time process is a Poisson process. The the arrival rate $\lambda(\tau)$ and conditional mark density $p(\alpha|\tau)$ can be defined arbitrarily. We shall denote the second and fourth moments of the conditional mark density as $\sigma_\alpha^2(\tau)$ and $\kappa_\alpha(\tau)$, respectively.

The expectation of the temporal moments are readily obtained by (6). To compute the covariance structure, we first note that the second order factorial intensity function of a Poisson point proceses is [12, 14],

$$\lambda^{(2)}(\tau, \tau') = \lambda(\tau)\lambda(\tau'). \tag{11}$$

Since the marks are independent, we have

$$A(\tau, \tau') = \begin{cases} \kappa_\alpha(\tau), & \tau = \tau' \\ \sigma_\alpha^2(\tau)\sigma_\alpha^2(\tau'), & \tau \neq \tau' \end{cases} \tag{12}$$

Using this in (7)–(10) gives

$$\sigma_{ij} = \int \kappa_\alpha(\tau) \lambda(\tau) \tau^{i+j} d\tau \tag{13}$$

which can be readily solved by analytical or numerical integration upon specification of $\kappa$ and $\lambda$. It should be noted, however, that the integral may diverge for some model settings and temporal moments. Then the covariances are not defined.

We remark that distinct settings with the same power delay spectra, and thus mean temporal moments, may lead to very different covariance structures. In fact, to give identical mean and covariance structures, both the power delay spectrum and the fourth moment delay spectrum $\kappa_\alpha(\tau)\lambda(\tau)$ should agree. This confirms the observations from [15, 2] where it was observed that models with the same power delay spectrum, but different higher moment spectra, produced different distribution of temporal moments.

### 4.1 Example with Exponential Power Decay and Gaussian Marks

We further specialise the model by considering in the room electromagnetic model in [2]. In this model, the transmitter and receiver are considered to be both in the same room. This setup yields power delay spectrum of the form

$$P(t) = G_0 \exp(-t/T), \quad t > 0, \tag{14}$$

where $G_0$ and $T$ are the reverberation gain and time, respectively. For the arrival rate we use the parametric function proposed in [2]

$$\lambda(t) = at^b, \quad t > 0 \tag{15}$$

**Table 1.** Simulation Settings

| Parameter | Value |
|---|---|
| Room dim. | $5 \times 5 \times 3\,\mathrm{m}^3$ |
| $g$ | 0.6 |
| $c$ | $3 \cdot 10^8\,\mathrm{m/s}$ |
| $t_{\max}$ | 120 ns |
| No. Monte Carlo runs | $10^4$ |

controlled by the parameters $a > 0$ and $b$. The mark distribution $p(\alpha|\tau)$ is chosen to be a circular complex Gaussian with conditional second moment

$$\sigma_\alpha^2(\tau) = \frac{P(\tau)}{\lambda(\tau)}, \quad t > 0. \tag{16}$$

Thus the fourth moment reads

$$\kappa(\tau) = 2\sigma_\alpha^4(\tau). \tag{17}$$

Inserting this model into (6) and (13) and integrating gives closed form expressions for the means and covariances

$$\mu_i = G_0 T^{i+1} i! \tag{18}$$

and

$$\sigma_{ij} = \frac{2G_o^2}{a}\left(\frac{T}{2}\right)^{i+j-b}\Gamma(i+j-b+1), \tag{19}$$

where the covariance expression is valid provided that $i+j-b+1 > 0$. For $i+j-b+1 \leq 0$, the integral diverges and the covariance does not exist. For integer $b$, the gamma function reduces as $\Gamma(i+j-b+1) = (i+j)!$.

Som observations follows from the (19). Firstly, the covariance $\sigma_{ij}$ is inversely proportional to any scaling of the arrival rate. Thus for models with smaller arrival rate the temporal moments tend to have stronger correlation. In contrast, the mean of any temporal moment is unaffected by a such a scaling. Secondly, the covariances depend on the reverberation time. Thirdly, the parameter $b$ which determines the growth of the arrival rate with delay, affects the covariance structure. In particular, if $b$ is large, the covariances for lower order temporal moments may not be finite. The is a practical concern in room electromagnetic models such as [2] for which $b = 2$. In this particular case, the distribution of $m_0$ is heavy tailed with undefined variance.

## 5 Simulation

We compare the theoretical to Monte Carlo simulations for the example in Subsection 4.1. The values for the model parameters, are set according to room electromagnetics:

$$G_0 = \frac{4\pi c}{V}, \quad T = -\frac{4V}{cS\ln g}, \quad a = \frac{4\pi c^3\omega^2}{V}, \quad b = 2. \tag{20}$$

where $V$ denotes the room volume, $S$ is the surface area of the room, $c$ is the speed of light and $g$ is a average reflection

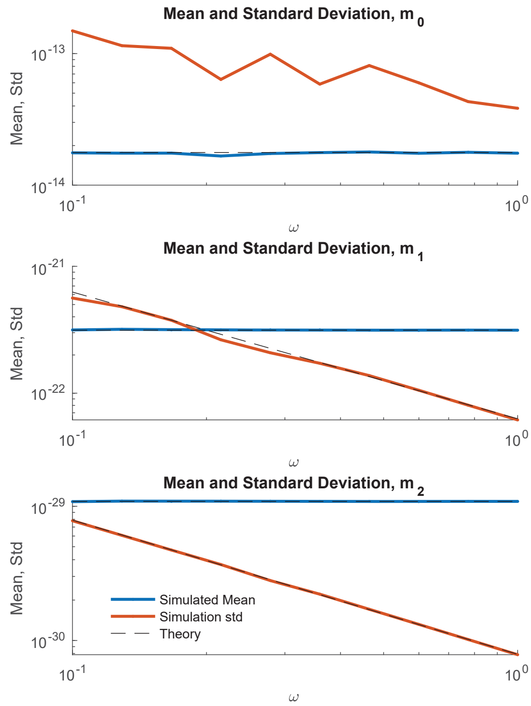


**Figure 1.** Mean and standard deviation of temporal moments. The simulation settings are given in Table 1. Channels are simulated op to a maximum time $t_{\max}$.

gain. The beam coverage fraction $\omega$ is a parameter which describes the directivity of an antenna and ranges from zero (infinitely directive) to one (isotropic). Both antennas of the link to have the same $\omega$. The directivity of the antenna scale the arrival rate, but no the power delay spectrum [15, 2].

The empirical mean and standard deviations are plotted in Figure 1 agrees with the theoretical results are when these exist. As predicted, the means are constant while the standard deviations $\sqrt{\sigma_{11}}$ and $\sqrt{\sigma_{22}}$ decay linearly with $\omega$. Figure 2 reports the empirical cdf and kernel density estimates of the distributions of $m_0, m_1$, and $m_2$. Moreover, lognormal distributions are shown for $m_0$ and $m_1$ for which the means and variances can be obtained. It appears that the lognormal distributions are quite accurate, and nearly perfect for the second order temporal moment.

## 6 Conclusion

The derived expression relates the covariance of temporal moments directly to the arrival rate and second-order intensity of the arrival process. This highlights the importance of the second order intensity in a stochastic channel model, an entity which is commonly ignored. By properly selecting the arrival process, both the mean and covariance of the temporal moments can be fitted to measurements. The results are applicable to any stochastic model for which the

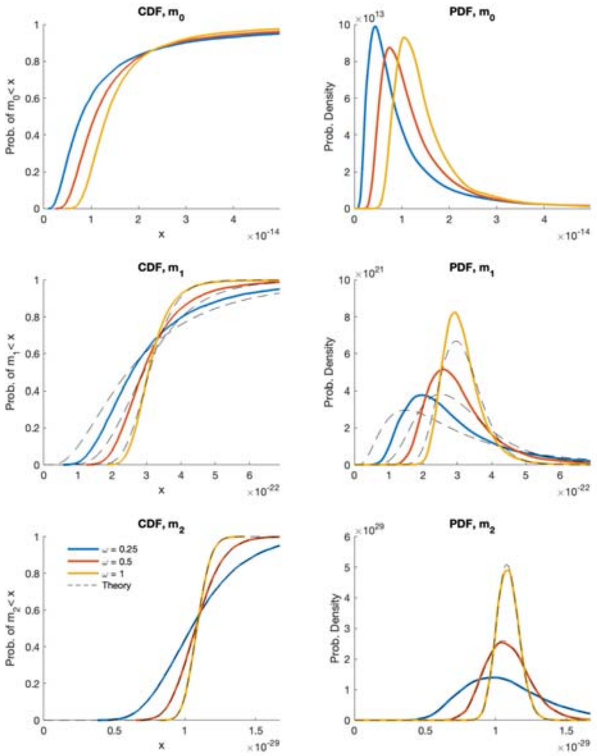


**Figure 2.** Empirical distribution of temporal moments with $\omega$ as parameter. For $m_1$ and $m_2$ log-normal distributions are included with theoretical mean and variance. The settings are as in Figure 1.

second order factorial intensity function for the delay process can be obtained.

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