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Excitation of a Layered Sphere by Multiple Point-generated Primary Fields

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Abstract

Excitation of a layered spherical medium by N external point sources is considered. The exact solution of the direct scattering problem is derived by adopting an efficient superposition scheme. Low-frequency approximations and relevant numerical results are presented. Extensions to inverse problems are pointed out.

1 Introduction

In this paper, we consider the boundary-value problem concerning the excitation of a layered spherical medium by N point sources, arbitrarily located at the scatterer's exterior. Scattering problems due to excitation by multiple sources have various applications, including the excitation of the human brain by the neurons currents [1], microphone array methods used in aeroacoustics and speech recognition [2], [3], cancer-treatment techniques [4] as well as sonar imaging used in oceanography [5].

The exact solution of the direct problem is determined by introducing an *overall superposition method*, which is a generalization of the T-Matrix method [6]-[8]. We make essential use of the distinction between *individual fields* (fields generated by a single point source) and *overall fields* (fields generated by a group of sources); this is particularly important for problems involving more than one sources exciting a scatterer [9]. Then, we define certain *excitation operators* and expand the overall primary and secondary fields in forms similar to that of an individual field. In this way, we reduce the problem of calculating the coefficients of the overall and individual secondary fields into a standard T-Matrix approach, as in the case of a single point source exciting the scatterer; the single source approach is presented e.g. in [6]. The proposed method has the advantage that it derives the coefficients of the *overall secondary field* as a sum of the coefficients of the individual secondary fields and requires only one application of the standard algorithm. Besides, from the expression of the overall coefficients, we can swiftly extract the coefficients of the individual secondary fields with no additional calculations.

Low-frequency approximations of the derived exact results are also obtained and related asymptotic expansions of the far-field patterns and scattering cross sections are presented. Such expansions can be efficiently utilized in inverse problems.

2 Mathematical Formulation

We consider a spherical scatterer of radius a_1, divided into P nested, concentric spherical shells V_p $(p = 1,\dots,P)$, by $P-1$ spherical surfaces each of radius a_p, with $p = 2,\dots,P$; see Fig. 1. Each layer V_p, defined by $a_{p+1} < r < a_p$, is characterized by wavenumbers k_p and mass densities ρ_p, with $p = 1,\dots,P-1$. The exterior V_0 of the scatterer is characterized by wavenumber k_0 and mass density ρ_0.

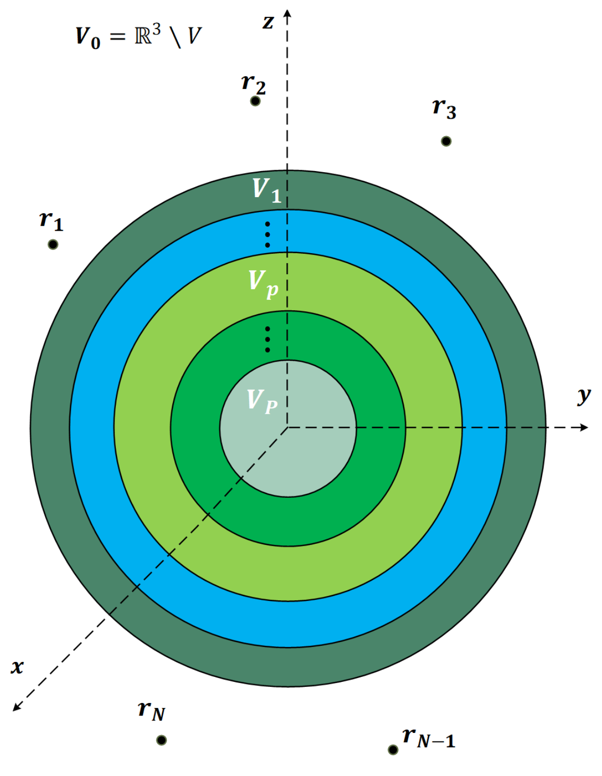

Figure 1. Layered spherical medium excited by N arbitrarily located external point sources

The layered scatterer is excited by N point sources located at $\mathbf{r}_j$ of V_0 for $j = 1,\dots,N$. These point sources emit spherical waves, with *individual primary fields* given by

$$u^{\text{pr}}(\mathbf{r};\mathbf{r}_j) = A_j \frac{\exp(\mathrm{i}k_0|\mathbf{r}-\mathbf{r}_j|)}{|\mathbf{r}-\mathbf{r}_j|}, \ \mathbf{r} \neq \mathbf{r}_j. \quad (1)$$

Each individual primary field interacts with the scatterer, generating *individual secondary* fields in V_0, which are denoted by $u^{\text{sec}}(\mathbf{r};\mathbf{r}_j)$. The *individual total field* in V_0 due to a source at $\mathbf{r}_j$ is denoted by $u^0(\mathbf{r};\mathbf{r}_j)$. According to Sommerfeld's method [10] (scattering superposition method [11]), it holds

$$u^0(\mathbf{r};\mathbf{r}_j) = u^{\rm pr}(\mathbf{r};\mathbf{r}_j) + u^{\rm sec}(\mathbf{r};\mathbf{r}_j). \quad (2)$$

Additionally, all individual total fields in V_0 satisfy the Sommerfeld's radiation condition.

The superposition of all individual primary fields will be called the *overall primary field* and denoted by $u^{\rm pr}(\mathbf{r};\mathbf{r}_1,\ldots,\mathbf{r}_N)$. The superposition of all individual total fields in V_p will be called the *overall field* of layer V_p and denoted by $u^p(\mathbf{r};\mathbf{r}_1,\ldots,\mathbf{r}_N)$.

On the boundaries of each layer V_p, all total individual and overall fields satisfy for $p = 1,\ldots,P-1$

$$u^{p-1}(\mathbf{r};\cdot) = u^p(\mathbf{r};\cdot), \quad r = a_p \quad (3)$$

$$\frac{1}{\rho_{p-1}}\frac{\partial u^{p-1}(\mathbf{r};\cdot)}{\partial r} = \frac{1}{\rho_p}\frac{\partial u^p(\mathbf{r};\cdot)}{\partial r}, \quad r = a_p. \quad (4)$$

As it is evident, the overall field of V_0 also satisfies the Sommerfeld's radiation condition. The medium's core V_P can be soft, hard or penetrable. For a soft or hard core, the respective boundary conditions read

$$u^{P-1}(\mathbf{r};\cdot) = 0, \quad r = a_P \quad (5)$$

$$\frac{\partial u^{P-1}(\mathbf{r};\cdot)}{\partial r} = 0, \quad r = a_P, \quad (6)$$

whereas for a penetrable core, conditions (3)-(4) hold for V_P as well.

The *individual far-field* due to a source located at $\mathbf{r}_j$ is denoted by $g_j(\hat{\mathbf{r}})$ and is defined by

$$u^{\rm sec}(\mathbf{r};\mathbf{r}_j) = g_j(\hat{\mathbf{r}})h_0(k_0r) + O(r^2), \quad r\to\infty, \quad (7)$$

where h_0 is the 0-order spherical Hankel function of the first kind. The superposition of all individual far-fields will be called *overall far-field* and denoted by $g(\hat{\mathbf{r}})$. Hence, it holds

$$u^{\rm sec}(\mathbf{r};\mathbf{r}_1,\ldots,\mathbf{r}_N) = g(\hat{\mathbf{r}})h_0(k_0r) + O(r^2), \quad r\to\infty. \quad (8)$$

Individual and *overall scattering cross sections* will be denoted, respectively, by σ_j and σ. They are defined by means of their corresponding far-fields as follows

$$\sigma_j = \frac{1}{k_0^2}\int_{S^2}|g_j(\hat{\mathbf{r}})|^2 {\rm d}s(\hat{\mathbf{r}}), \quad (9)$$

$$\sigma = \frac{1}{k_0^2}\int_{S^2}|g(\hat{\mathbf{r}})|^2 {\rm d}s(\hat{\mathbf{r}}). \quad (10)$$

3 Excitation Operators and Field Expansions

Choosing the coordinate system (r,θ,ϕ) so that the origin is located at the sphere's center O, each point source is located at $\mathbf{r}_j = (r_j,\theta_j,\phi_j)$ with $r_j > a_1$, for $j = 1,\ldots,N$. The individual primary fields are given by [8]

$$u_0^{\rm pr}(\mathbf{r};\mathbf{r}_j) = 4\pi {\rm i}k_0A_j \begin{cases} \sum_{n=0}^{\infty}\sum_{m=-n}^{n}(-1)^m Y_n^{-m}(\hat{\mathbf{r}}_j)Y_n^m(\hat{\mathbf{r}}) \\ h_n(k_0r)j_n(k_0r_j), \ r > r_j \\ \sum_{n=0}^{\infty}\sum_{m=-n}^{n}(-1)^m Y_n^{m}(\hat{\mathbf{r}}_j)Y_n^{-m}(\hat{\mathbf{r}}) \\ j_n(k_0r)h_n(k_0r_j), \ r < r_j, \end{cases} \quad (11)$$

while the individual secondary fields in V_p are expanded as

$$u^p(\mathbf{r};\mathbf{r}_j) = 4\pi {\rm i}k_0A_j\sum_{n=0}^{\infty}\sum_{m=-n}^{n}(-1)^m Y_n^{-m}(\hat{\mathbf{r}}_j)Y_n^m(\hat{\mathbf{r}}) \\ h_n(k_0r_j)(a_{j,n}^p j_n(k_pr) + b_{j,n}^p h_n(k_pr)), \quad (12)$$

where j_n and h_n are the n-th order spherical Bessel and Hankel functions, respectively. Functions Y_n^m and Y_n^{-m} denote the spherical harmonic functions.

Now, we define the following *excitation operators*

$$\mathscr{J}_{n,m}(\mathbf{x}) = \sum_{j=1}^{N} A_jY_n^{-m}(\hat{\mathbf{r}}_j)j_n(k_0r_j){\rm x}_j, \quad (13)$$

$$\mathscr{H}_{n,m}^1(\mathbf{x}) = \sum_{j=1}^{N} A_jY_n^{m}(\hat{\mathbf{r}}_j)h_n(k_0r_j){\rm x}_j, \quad (14)$$

$$\mathscr{H}_{n,m}^2(\mathbf{x}) = \sum_{j=1}^{N} A_jY_n^{-m}(\hat{\mathbf{r}}_j)h_n(k_0r_j){\rm x}_j, \quad (15)$$

where $\mathbf{x} = ({\rm x}_1,\ldots,{\rm x}_N)$ is an arbitrary vector or $\mathbb{R}^N$. We denote $\mathscr{A}_{n,m}^p = \mathscr{H}_{n,m}^2(\mathbf{a}_n^p)$ and $\mathscr{B}_{n,m}^p = \mathscr{H}_{n,m}^2(\mathbf{b}_n^p)$, where $\mathbf{a}_n^p = (a_{1,n}^p,\ldots,a_{N,n}^p)$ and $\mathbf{b}_n^p = (b_{1,n}^p,\ldots,b_{N,n}^p)$ are the vectors with components the unknown coefficients of the individual secondary fields, take under consideration expansions (11), (12), and utilize the definitions of overall primary and secondary fields. In this way, we obtain the following expansions for the overall primary field

$$u^{\rm pr}(\mathbf{r};\mathbf{r}_1,\ldots,\mathbf{r}_N) = 4\pi {\rm i}k_0 \begin{cases} \sum_{n=0}^{\infty}\sum_{m=-n}^{n}(-1)^m Y_n^{m}(\hat{\mathbf{r}}) \\ h_n(k_0r)\mathscr{J}_{n,m}(\mathbf{q}), \quad r > \max[r_j] \\ \vdots \\ \sum_{n=0}^{\infty}\sum_{m=-n}^{n}(-1)^m Y_n^{-m}(\hat{\mathbf{r}}) \\ j_n(k_0r)\mathscr{H}_{n,m}^1(\mathbf{q}), \quad r < \min[r_j], \end{cases} \quad (16)$$

where $\mathbf{q}$ denotes the N-dimensional vector $(1,1,\ldots,1)$. Similarly, the overall secondary field of V_p has the expansion

$$u^p(\mathbf{r};\mathbf{r}_1,\ldots,\mathbf{r}_N) = 4\pi {\rm i}k_0\sum_{n=0}^{\infty}\sum_{m=-n}^{n}(-1)^m Y_n^m(\hat{\mathbf{r}}) \\ \Big(\mathscr{A}_{n,m}^p j_n(k_pr) + \mathscr{B}_{n,m}^p h_n(k_pr)\Big). \quad (17)$$

4 Exact Solution of the Direct Problem

Considering properties of the harmonic functions [12], and imposing boundary conditions, on the boundaries of layers

V_p for $p = 2,\ldots,P-1$, we obtain

$$\begin{bmatrix}\mathscr{A}^p_{n,m}\\ \mathscr{B}^p_{n,m}\end{bmatrix} = \mathbf{T}_n^{(1\to p)}\begin{bmatrix}\mathscr{A}^1_{n,m}\\ \mathscr{B}^1_{n,m}\end{bmatrix}, \quad (18)$$

where $\mathbf{T}_n^p$ is the *transition matrix* from layer V_{p-1} to layer V_p (see [6]), and $\mathbf{T}_n^{(1\to p)}$ is the transition matrix from layer V_1 to layer V_p given by $\mathbf{T}_n^{(1\to p)} = \mathbf{T}_n^p\mathbf{T}_n^{p-1}\ldots\mathbf{T}_n^2$. Particularly, for the boundary of layer V_1 we have

$$\begin{bmatrix}\mathscr{A}^1_{n,m}\\ \mathscr{B}^1_{n,m}\end{bmatrix} = \mathbf{T}_n^1\begin{bmatrix}\mathscr{H}^1_{n,m}(\mathbf{q})\\ \mathscr{B}^0_{n,m}\end{bmatrix}. \quad (19)$$

Combining (18) and (19), we obtain

$$\begin{bmatrix}\mathscr{A}^{P-1}_{n,m}\\ \mathscr{B}^{P-1}_{n,m}\end{bmatrix} = \mathbf{T}_n^{(0\to P-1)}\begin{bmatrix}\mathscr{H}^1_{n,m}(\mathbf{q})\\ \mathscr{B}^0_{n,m}\end{bmatrix}. \quad (20)$$

Depending on the conditions on the core's boundary, we can extract the unknown coefficients of the overall secondary field. For a soft or hard core, we obtain

$$\mathscr{B}^0_{n,m} = -\frac{\Psi_n^1(k_{P-1}a_P)\,\mathscr{H}^1_{n,m}(\mathbf{q})}{\Psi_n^2(k_{P-1}a_P)}, \quad (21)$$

where $\Psi_n^i(x)$ with $i = 1,2$ denotes the i component of the *boundary transition vector*

$$\mathbf{\Psi}_n(x) = \Big[f_n(x)\;\; g_n(x)\Big]\,\mathbf{T}_n^{(0\to P-1)}. \quad (22)$$

The exact form of f_n, g_n depends on the boundary conditions, e.g.

$$f_n(x) = \begin{cases} j_n(x), & \text{soft core}\\ j_n'(x), & \text{hard core}\end{cases} \quad (23)$$

$$g_n(x) = \begin{cases} h_n(x), & \text{soft core}\\ h_n'(x), & \text{hard core}\end{cases} \quad (24)$$

On the other hand, for a penetrable core, we have

$$\mathscr{B}^0_{n,m} = -\frac{T^{(0\to P)}_{21,n}\,\mathscr{H}^1_{n,m}(\mathbf{q})}{T^{(0\to P)}_{22,n}}, \quad (25)$$

where $T^{(0\to P)}_{ij,n}$ denotes the ij element of transition matrix $\mathbf{T}_n^{(0\to P)}$. The coefficients of the individual secondary fields can be obtained directly from (21), (25) as follows

$$b^0_{j,n} = -\frac{\Psi_n^1(k_{P-1}a_P)\,\mathscr{H}^1_{n,m}(\mathbf{h}_j)}{\Psi_n^2(k_{P-1}a_P)}, \quad (26)$$

$$b^0_{j,n} = -\frac{T^{(0\to P)}_{21,n}\,\mathscr{H}^1_{n,m}(\mathbf{h}_j)}{T^{(0\to P)}_{22,n}}, \quad (27)$$

where

$$\mathbf{h}_j = \frac{\mathbf{e}_j}{A_j Y_n^{-m}(\hat{\mathbf{r}}_j)h_n(k_0 r_j)}, \quad (28)$$

with $\mathbf{e}_j$ the vectors of the standard basis of $\mathbb{R}^N$. The overall far-field $g(\hat{\mathbf{r}})$ and the overall scattering cross section σ take the forms, respectively

$$g(\hat{\mathbf{r}}) = 4\pi\mathrm{i}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}(-1)^m(-\mathrm{i})^n Y_n^m(\hat{\mathbf{r}})\mathscr{B}^0_{n,m}, \quad (29)$$

$$\sigma = \frac{4\pi}{k_0^2}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}(2n+1)\frac{(n-m)!}{(n+m)!}\left|\mathscr{B}^0_{n,m}\right|^2. \quad (30)$$

Eq. (26) coincides with (11) of [8]. For $\theta_j = 0$, Eq. (27) reduces to (3.10) of [6].

5 Low Frequency Approximations

Next, we provide approximations for the overall far-field and the overall scattering cross section, in the case where a 2-layered sphere with a soft core is excited by N point sources. Denoting $\tau_j = a_1/r_j$, $\rho = \rho_1/\rho_0$, $\eta = k_1/k_0$, $\xi = a_1/a_2$ and assuming that $A_j = r_j e^{-ik_0 r_j}$, by means of (29), we obtain the expansion of the overall far-field

$$\begin{aligned}
g(\hat{\mathbf{r}}) = \sum_{j=1}^{N} e^{-\kappa/\tau_j}\Bigg\{&\kappa S_1 + \\
&\kappa^2\Big[\rho\eta^2(S_1)^2 + S_2\tau_j\Big(\cos\theta\cos\theta_j + \\
&\sin\theta\sin\theta_j\cos(\phi_j-\phi)\Big)\Big] + \\
&\kappa^3\Big[S_3 - S_2\Big(\cos\theta\cos\theta_j + \sin\theta\sin\theta_j\cos(\phi_j-\phi)\Big) + \\
&\Big(\sin2\theta\sin2\theta_j\cos(\phi_j-\phi) + \sin^2\theta\sin^2\theta_j\cos(2(\phi_j-\phi)) + \\
&\frac{1}{3}(3\cos^2\theta_j-1)(3\cos^2\theta-1)\Big)\frac{\tau_j^2}{4}S_4\Big]\Bigg\} + O(\kappa^4). \quad (31)
\end{aligned}$$

For the overall scattering cross section, utilizing (30), we arrive at

$$\begin{aligned}
\sigma = 4\pi a_1^2\Bigg\{&(S_1)^2\left|\sum_{j=1}^{N} e^{-\kappa/\tau_j}\right|^2 \\
&\left[1-(k_0a_1)^2(S_1)^2\frac{\rho\eta^2}{\xi}(\rho\xi+2-2\rho)\right] + \\
&(k_0a_1)^2\frac{(S_2)^2}{3}\left|\sum_{j=1}^{N}\tau_j e^{-\kappa/\tau_j}\right|^2\Bigg\} + O(\kappa^4), \quad (32)
\end{aligned}$$

where

$$S_1 = \frac{1}{\rho-1-\xi\rho}, \quad (33)$$

$$S_2 = \frac{\xi^3(1-\rho)+2+\rho}{\xi^3(1+2\rho)+2-2\rho}, \quad (34)$$

$$S_3 = \frac{\rho\eta^2(2\xi\rho+\rho-1)}{3\xi}(S_1)^3, \quad (35)$$

$$S_4 = -\frac{2\xi^5(1-\rho)+3+2\rho}{2\xi^5(2+3\rho)+3-3\rho}. \quad (36)$$

Eqs. (31) and (32) can be efficiently utilized in the development of inverse source and inverse medium algorithms.

6 Numerical Results

We show numerical results for the comparisons between the exact overall cross section and its corresponding low-frequency approximation for the case of a 2-layered sphere with a soft core. The sphere is excited by two point sources located at $r_1 = 1.3a_1$ and $r_2 = 1.7a_1$. In the first case, we suppose that the layer V_1 has physical parameters $\eta = 1.75$ and $\rho = 1.5$, whereas in the second case $\eta = 2.25$ and $\rho = 2.5$. For the computation of the exact overall cross section, we utilized formula (30).

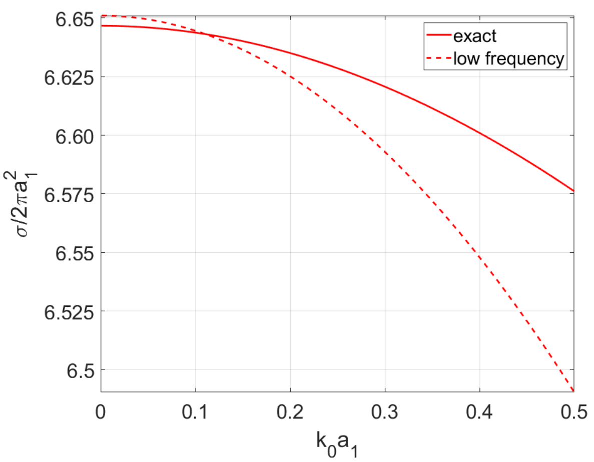

Figure 2. Comparison between the exact cross section and its low frequency approximation for a 2-layered sphere with a soft core excited by $N = 2$ external point sources for physical parameters $\eta = 1.75$ and $\rho = 1.5$.

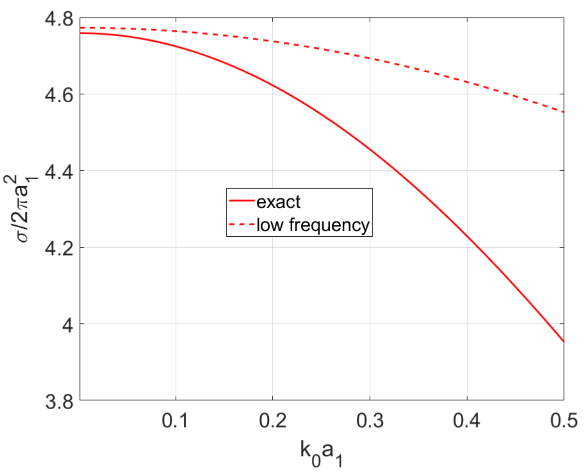

Figure 3. As in Fig. 2, but for physical parameters, $\eta = 2.25$ and $\rho = 2.5$.

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