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Complex Waves in a Partially Shielded Dielectric Layer

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Abstract

The propagation of monochromatic TE-polarized waves in a partially shielded dielectric layer is considered. The existence of infinitely many complex leaky waves is proved as well as the absence of complex surface waves.

1 Introduction

The dielectric layer (DL) is one of the most well-studied waveguide structures in electromagnetics [1, 2, 3, 4]. In fact, DL is the simplest plane-parallel waveguide (from the geometrical point of view) and its dispersion equation (DE) can be written explicitly. On the other hand such a structure is widely used in practice (planar optical waveguides). However, still, there is no rigorous proof of the presence (or absence) of infinitely many real or complex eigenwaves propagating in DL, to the best of our knowledge.

The main attention of this study is paid to the analysis of the surface and leaky waves in a partially shielded DL (PSDL). Note that the closed-form DE of DL can easily be solved numerically (for real propagation constants). A large number of numerical results and calculations of surface waves in DL were obtained [5, 6, 7], however without completing the justification of the method including rigorous proofs of the existence of the DE roots. The existence of a finite number of surface propagating waves is well known and has been proved graphically [8] (note that here a mathematically rigorous proof has never been completed). For leaky waves (including complex ones), the solution of DE on a two-sheet Riemann surface was considered in [9]. In this paper, we focus on theoretical analysis rather than on numerical results and numerical methods ending up with mathematically correct proofs. The main purpose of this article is not only the correct verification of the occurrence of complex waves in a DL but also a rigorous proof of the existence of an infinite number of complex leaky waves. We will look for the odd TE modes of DL [2, 3]. For even TE-waves, as well as for odd and even TM-waves, the same results hold.

The general results about the existence of propagating waves in nonhomogeneous waveguides and localization of propagation constants on the complex plane have been recently obtained in [10, 11, 12, 13].

2 Statement of the problem

Consider the three-dimensional half space $\mathbb{R}^3$ equipped with the Cartesian coordinate system $Oxyz$ and filled with an isotropic source-free medium having permittivity $\varepsilon_0\varepsilon_2 \equiv const$ and permeability $\mu_0 \equiv const$, where ε_0 and μ_0 are permittivity and permeability of vacuum. We consider electromagnetic waves propagating through a dielectric layer located between two halfspaces $x < 0$ and $x > h$:

$$\Sigma := \{(x,y,z) : 0 \leqslant x \leqslant h\}.$$

The boundary $x = 0$ is the projection of the surface of perfectly conducting (PEC) shield and $x = h$ is the projection of the dielectric surface. The geometry of the problem is shown in Fig. 1.

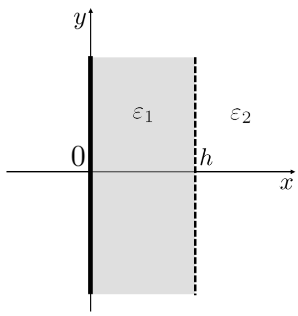

Figure 1. Geometry of the problem.

Determination of normal TE-polarized waves reduces to finding nontrivial running-wave solutions of the homogeneous system of Maxwell equations depending on the coordinate z along which the structure is regular in the form $e^{i\gamma z}$,

$$\begin{cases} \nabla \times \mathbf{H} = -i\tilde{\varepsilon}\mathbf{E}, \\ \nabla \times \mathbf{E} = i\mathbf{H}, \end{cases} \tag{1}$$

$$\mathbf{E} = \left(0,\, E_y(x)e^{i\gamma z},\, 0\right),\ \mathbf{H} = \left(H_x(x)e^{i\gamma z},\, 0,\, H_z(x)e^{i\gamma z}\right),$$

with the boundary conditions for the tangential electric component on the PEC surface ($x = 0$)

$$E_y(0) = 0, \tag{2}$$

transmission conditions for the tangential electric and magnetic field components on the permittivity discontinuity surface ($x = h$)

$$[E_y]\big|_{x=h} = 0,\ [H_z]\big|_{x=h} = 0, \tag{3}$$

where $[f]|_{x_0} = \lim_{x \to x_0 - 0} f(x) - \lim_{x \to x_0 + 0} f(x)$; and the radiation condition at infinity which will be formulated and discussed later.

The Maxwell system (1) is written in the normalized form. The passage to dimensionless variables has been carried out; namely, $k_0 x \to x$, $\gamma \to \gamma/k_0$, $\sqrt{\mu_0/\varepsilon_0}\mathbf{H} \to \mathbf{H}$, $\mathbf{E} \to \mathbf{E}$, where $k_0^2 = \omega^2 \varepsilon_0 \mu_0$, ω is a circular frequency (the time factor $e^{-i\omega t}$ is omitted everywhere).

We assume that the relative permittivity in the entire space have the form

$$\widetilde{\varepsilon} = \begin{cases} \varepsilon_1, \ 0 \leq x \leq h, \\ \varepsilon_2, \quad\quad x > h, \end{cases} \tag{4}$$

and that $\varepsilon_1 > \varepsilon_2$.

The problem on normal waves is an eigenvalue problem for the Maxwell equations with spectral parameter γ which is the wave normalized propagation constant.

The normal wave field in the waveguide can be represented using one scalar function

$$u := E_y(x). \tag{5}$$

Thus, the problem is reduced to finding tangential component u of the electric field. Throughout the text below, $(\,\cdot\,)'$ stands for differentiation with respect to x.

We have the following eigenvalue problem for the tangential electric field component u: find $\gamma \in \mathbb{C}$ such that there exist nontrivial solutions of the differential equation

$$u'' + \left(\widetilde{\varepsilon} - \gamma^2\right) u = 0, \ x > 0, \tag{6}$$

satisfying the boundary condition for $x = 0$

$$u(0) = 0, \tag{7}$$

transmission conditions for $x = h$

$$[u]|_h = 0, \ \left[u'\right]\big|_h = 0, \tag{8}$$

and the condition at infinity.

Thus the resulting field $(\mathbf{E}, \mathbf{H})$ will satisfy all conditions (1)–(3).

Definition 1. The *propagating* wave is characterised by real parameter γ.

Definition 2. The *evanescent* wave is characterised by pure imaginary parameter γ.

Definition 3. The *complex* wave is characterised by complex parameter γ such that $\operatorname{Re}\gamma \operatorname{Im}\gamma \neq 0$.

Definition 4. The *surface* wave is such that $u(x) \to 0$, $x \to \infty$.

Definition 5. The *leaky* wave is such that $u(x) \to \infty$, $x \to \infty$.

Remark 1. Propagation constant γ characterises the behaviour of a wave (propagating, evanescent, or complex) in the z-direction. Classification of waves as surface or leaky depends on the behaviour in x-direction. Thus, a wave can increase in one direction and decrease in the other direction.

For $0 < x < h$, we have $\widetilde{\varepsilon} = \varepsilon_1$; thus from (6) we obtain the equation

$$u'' + \lambda^2 u = 0, \tag{9}$$

where

$$\lambda^2 = \varepsilon_1 - \gamma^2, \tag{10}$$

and λ is a new (complex) spectral parameter.

In view of the boundary condition for the tangential electric field component on the PEC surface (7), we obtain a solution of this equation in the form

$$u(x;\lambda) = C_1 \sin \lambda x, \ 0 < x < h, \tag{11}$$

where C_1 is a constant.

For $x > h$, we have $\widetilde{\varepsilon} = \varepsilon_2$; then from (6) we obtain the equation

$$u'' - \left(\varepsilon^2 - \lambda^2\right) u = 0, \tag{12}$$

where $\varepsilon^2 = \varepsilon_1 - \varepsilon_2 > 0$. We choose a solution of this equation in the form

$$u(x;\lambda) = C_2 e^{-(x-h)\sqrt{\varepsilon^2 - \lambda^2}}, \ x > h, \tag{13}$$

where C_2 is a constant.

Remark 2. In the general case we have the solution of equation (12) in the form

$$u = C_2 e^{-(x-h)\sqrt{\varepsilon^2 - \lambda^2}} + C_3 e^{(x-h)\sqrt{\varepsilon^2 - \lambda^2}}, \ x > h, \tag{14}$$

where C_2 and C_3 are arbitrary constants. Below we explain our choice of the form (13).

Note that the study of the waves determined by solution (13) was performed earlier (see, e.g. [4]). However, a rigorous proof of the existence of complex and leaky waves has not been completed, as well as the classification of waves.

From transmission conditions (8) and solutions (11) and (13) we obtain the DE

$$\tan \lambda h + \frac{\lambda}{\sqrt{\varepsilon^2 - \lambda^2}} = 0. \tag{15}$$

In (15) the square root is a two-valued complex function and we will specify its branch below.

Solutions λ of equation (15) define several different types of waves.

DE (15) and its real- and imaginary-wavenumber solutions λ for PSDL are well-studied and can be found in many textbooks on electromagnetics, e.g. in [14].

3 Surface waves

In this section we will consider surface waves for which the electromagnetic field decays at $x \to \infty$. We suppose that

$$\mathrm{Re}\sqrt{\varepsilon^2-\lambda^2} > 0 \tag{16}$$

in order to specify the radiation condition at infinity. This condition determines surface waves, i.e. the waves decaying at infinity according to (13). Under condition (16) we obtain that $C_3 = 0$ in (14) for the surface waves.

Theorem 1. *Equation* (15) *under condition* (16) *has no (complex) solutions.*

Proof. Rewrite equation (15) as follows

$$\tan\lambda h = -\frac{\lambda}{{}_{+}\sqrt{\varepsilon^2-\lambda^2}}; \tag{17}$$

taking the squares from the right and left sides we get

$$\sin^2\lambda h = \lambda^2\varepsilon^{-2},$$

hence,

$$(\varepsilon\sin\lambda h - \lambda) = 0 \text{ or } (\varepsilon\sin\lambda h + \lambda) = 0. \tag{18}$$

Assume that the first equation of (18) has a solution $\lambda_m = \alpha_m + i\beta_m$. Then we have

$$\sin\lambda_m h = \lambda_m\varepsilon^{-1}$$

and from (17) we obtain

$$\mathrm{Re}\sqrt{\varepsilon^2-\lambda_m^2} = -\varepsilon\cos\alpha_m h\cosh\beta_m h > 0;$$

therefore $\cos\alpha_m h < 0$. On the other hand we have

$$\mathrm{Im}\sin\lambda_m h = \mathrm{Im}\,\lambda_m h/\varepsilon,$$

hence

$$\cos\alpha_m h = \frac{\beta_m h}{\varepsilon\sinh\beta_m h} > 0$$

which is a contradiction yielding a conclusion that the first equation (18) has no solution λ_m satisfying condition (16).

Repeating the above reasoning for the second equation (18) we arrive at the same conclusion. Thus equation (15) has no solution under condition (16). $\square$

Theorem 1 establishes that there are no complex surface waves satisfying condition (16).

4 Leaky waves

The occurrence and analysis of leaky waves together with their various applications in microwave engineering have been a subject of numerous studies since the early 1950s (see e.g.[15, 16, 17]). However, rigorous proofs of their existence has never been completed.

We assume that

$$\mathrm{Re}\sqrt{\varepsilon^2-\lambda^2} < 0 \tag{19}$$

in order to specify the radiation condition at infinity. This condition determines the leaky waves increasing at infinity. We assume that $C_3 = 0$ in (14) and solution u have the form (13).

Theorem 2. *Under condition* (19) *equation* (15) *has infinitely many (complex) roots forming a number sequence that tends to infinity.*

Proof. Introduce the function $f(z) := z\sin\left(z^{-1}\right)$ where $z = 1/\lambda h$. $f(z)$ has an isolated essential singularity at $z = 0$. Set $a := \varepsilon h > 0$ and consider two equations $f(z) = a^{-1}$ and $f(z) = -a^{-1}$. From Picard's theorem (see [18]) it follows that one of these equations has infinitely many roots $z_k (z_k \neq 0)$, $z_k \to \infty$, $k \to \infty$ in a neighborhood of the point $z = 0$. Indeed, from Picard's theorem we conclude that there is only one exceptional value A such that the equation $f(z) = A$ does not have infinitely many roots.

Let us consider the first equality $\sin\left(z^{-1}\right) = a^{-1}z^{-1}$. Then

$$\cos\left(z^{-1}\right) = \pm\sqrt{1-\sin^2\left(z^{-1}\right)} = \pm\sqrt{1-a^{-2}z^{-2}}. \tag{20}$$

Taking a real part of $\cos\left(z^{-1}\right)$, we get

$$\mathrm{Re}\cos\left(z^{-1}\right) = \cos\left(z'|z|^{-2}\right)\cosh\left(z''|z|^{-2}\right),\ z = z' + iz''$$

($z'' \neq 0$ because there are no real solutions of the equation $z\sin\left(z^{-1}\right) = a^{-1}$ for $|z| < \frac{1}{a}$). Since $\cosh\left(z''|z|^{-2}\right) > 0$, we obtain the following equality

$$\mathrm{sign}\,\mathrm{Re}\cos\left(z^{-1}\right) = \mathrm{sign}\cos\left(z'|z|^{-2}\right).$$

On the other hand, taking imaginary part of $\sin\left(z^{-1}\right)$ we have

$$\mathrm{Im}\sin\left(z^{-1}\right) = \cos\left(z'|z|^{-2}\right)\sinh\left(-z''|z|^{-2}\right) = -z''a|z|^{-2} \Rightarrow$$
$$\Rightarrow \cos\left(z'|z|^{-2}\right) = \frac{z''}{a|z|^2\sinh\left(z''|z|^{-2}\right)} > 0.$$

Consequently, $\mathrm{sign}\,\mathrm{Re}\cos\left(z^{-1}\right) > 0$. According to the condition (19) we have

$$\mathrm{Re}\sqrt{\varepsilon^2-\lambda^2} = \varepsilon h\,\mathrm{Re}\sqrt{1-a^{-2}z^{-2}} < 0,$$

i.e. we should choose the sign of the root in the formula (20) as follows

$$\cos\left(z^{-1}\right) = -\sqrt{1-a^{-2}z^{-2}}$$

in order to satisfy the condition $\operatorname{sign}\operatorname{Re}\cos\left(z^{-1}\right) > 0$.

Let z_k be a root of equation $f(z) = a^{-1}$; then

$$\sin\left(z_k^{-1}\right) = a^{-1}z_k^{-1},\ \cos\left(z_k^{-1}\right) = -\sqrt{1-a^{-2}z_k^{-2}}.$$

Hence

$$\tan\left(z_k^{-1}\right) + \frac{a^{-1}z_k^{-1}}{\sqrt{1-a^{-2}z_k^{-2}}} = 0. \tag{21}$$

Thus we obtain that the equation

$$\tan\lambda_k h + \frac{\lambda_k h}{\sqrt{\dfrac{\varepsilon^2}{h^2} - \lambda_k^2 h^2}} = 0,$$

has a solution $\lambda_k = z_k^{-1}h^{-1}$.

Repeating the above consideration for the equation $f(z) = -a^{-1}$ we arrive at the same conclusion.

Since one of the equations $f(z) = a^{-1}$ or $f(z) = -a^{-1}$ has infinitely many roots in the neighborhood of $z = 0$ we obtain that the equation (15) has infinite number of roots at infinity under condition (19). □

Theorem 2 establishes the existence of infinite number of complex leaky waves (that increase at infinity satisfying condition (19)). The propagation constants of such leaky waves are $\tilde{\gamma}_k^{leaky} = \sqrt{\varepsilon_1 - \lambda_k^2}$.

5 Conclusion

We have proved the existence of complex, leaky, and surface waves in PSDL. Two general wave families of the propagating surface and complex leaky waves have been identified governed by the dependence of their wavenumbers $\sqrt{\varepsilon_1 - \gamma^2}$ on the problem parameters, location on the complex plane, and behavior at infinity.

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