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Diffraction of acoustic waves from a point source over an impedance wedge

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Abstract

Contrary to the problem of scattering of electromagnetic waves from a Hertz dipole located over an impedance wedge, the corresponding acoutic problem is explicitly solvable (i.e. by quadrature). In this work we study diffraction of a spherical acoustic wave due to a point source, by an impedance wedge. In the exterior of the wedge the acoustic pressure satisfies the stationary wave (Helmholtz) equation and classical impedance boundary conditions on two faces of the wedge, as well as Meixner's condition at the edge and the radiation conditions at infinity. Solution of the boundary value problem is represented by a Weyl type integral and its asymptotic behavior is discussed. On this way, we derive various components in the far field interpreting them accordingly and discussing their physical meaning.

1 Introduction

Geometrical Theory of Diffraction (GTD) is one of the most powerful and widely exploited asymptotic theories applied in engineering and research practice. It is based on commonly accepted high frequency localization principle. As is well known, practical use of GTD in various ray-tracing procedures requires knowledge of diffraction or excitation coefficients which should be incorporated into such procedures. These coefficients are responsible for the local transformation of the wave field attributed to the rays interacting with some points of the boundary. It is remarkable, however, that these coefficients are traditionally determined from canonical problems locally describing the process of such interaction. Due to the high frequency localization principle various canonical problems are of importance in applications and, therefore, attract attention of researchers.

The canonical problem under consideration plays an important role in numerous applications including, first of all, Geometrical Theory of Diffraction and its various modifications. In the case of perfect wedges, i.e. for those with ideal boundary conditions, some results dealing the 3D diffraction of waves from a point source are known in the literature, see the references in [2]–[3]. To our knowledge, a basic idea is in use of the Weyl type integral representation [1] of solution for the corresponding boundary value problem. This representation is actually a plane wave expansion for the incident field from a point source and a similar representation for the field diffracted by the perfect wedge. Such an expansion is based on solution of the plane wave diffraction problem by a wedge for real angles of incidence which is then analytically continued for the complex-valued angles. An appropriate choice of integration contours leads to a rapidly convergent iterated integral further called Weyl integral representation. For a perfect wedge the integrand of Weyl representation has a simple elementary form and further analysis deals with asymptotic evaluation of the Weyl integral for large distances from the edge.

In the present report we make use of the possibility to solve the acoustic problem explicitly, i.e. the problem of diffraction of an acoustic plane wave which is skew incident at the edge of an impedance wedge. This solution discussed below is actually some strict modification of the Malyuzhinets solution and is found explicitly. Having Weyl integral representation of the wave field for the point source illumination of an impedance wedge, we evaluate the integral by means of the saddle point technique (or stationary phase technique). To this end, we deform the integration contours appropriately taking into account the corresponding singularities captured. Then we evaluate various terms asymptotically and discuss physical meaning of the wave components at far distances from the edge. These components are reflected waves from the faces, the space wave from the edge as well as the surface waves from the edge. The amplitudes and phases of these waves depend on position of the point source. Provided that the point source is in some close vicinity of a face of the wedge, additional components in the asymptotics arise. In this case the source additionally excites surface wave that also propagates along the face to the edge then it is reflected from and transmitted across the edge, giving rise to an additional space wave from the edge.

2 Formulation

An acoustic point source illuminates an impedance wedge of the opening angle 2Φ. The wave field interacts with the faces and with the edge giving rise to the scattered field which consists of various far-field components having clear physical meaning. The asymptotic description of these components in the far field is one of the main goals of our study.

In the exterior Ω of the wedge with the surface S consisting of two faces S_+ and S_- (Fig. 1) the acoustic wave field u

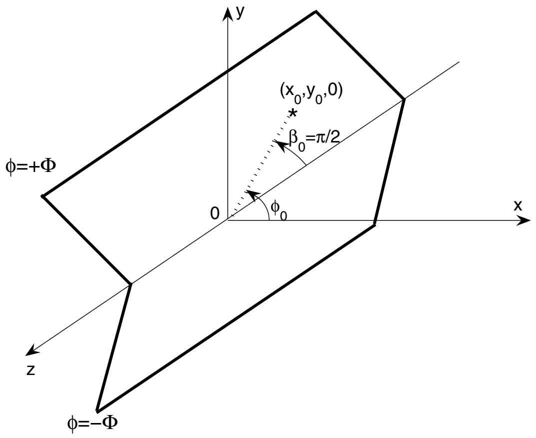

Figure 1: Point source over an impedance wedge

(Green's function) satisfies the Helmholtz equation

$$(\triangle + k^2)u(X,Y,Z) = -\delta(X - x_0)\delta(Y - y_0)\delta(Z)\,, \quad (1)$$

where the point source is located at $(x_0, y_0, 0)$. Instead of the Cartesian coordinates, it is also useful to introduce cylindrical coordinates r, φ, z,

$$X = r\cos\varphi,\ Y = r\sin\varphi,\ Z = z.$$

In these coordinates the position of the point source is given by $(r_0, \varphi_0, 0)$ and the domain Ω is $\Omega = \{(r, \varphi, z) : r > 0, |\varphi| < \Phi, |z| < \infty\}$.

The total acoustic field $u = u(r, \varphi, z)$ fulfills the impedance boundary conditions on the wedge's faces $S_\pm$

$$\left(\pm\frac{1}{r}\frac{\partial u}{\partial \varphi} - \mathrm{i}k\eta_\pm u\right)\Bigg|_{\varphi=\pm\Phi} = 0\,, \quad (2)$$

where $k > 0$, $\pi/2 < \Phi \le \pi$, $\eta_\pm$ are the surface impedances. It is worth commenting on the real and imaginary parts of the surface impedances. We assume that $\Re\eta_\pm = 0$, which means that the impedance surfaces are not absorbing, and $\Im\eta_\pm < 0$. In acoustics the latter restriction implies that the impedance surfaces $S_\pm$ can support surface waves. (The time dependence $\exp\{-\mathrm{i}\omega t\}$ is assumed and suppressed throughout the report.)

The wave field behavior at the edge satisfies Meixner's condition, as $r \to 0$,

$$u(r, \varphi, z) = C + O(r^\delta)\,,\ \delta > 0\,, \quad (3)$$

for arbitrary fixed z and uniformly w.r.t. φ. An appropriate radiation condition at infinity is implied. It can be traditionally formulated and added to (1)–(3).

3 Description of the results

We make use of the Weyl integral representation describing solution of the problem in an explicit form of triple integral. This report deals basicly with asymptotic evaluation of the Weyl integral and with description of the various

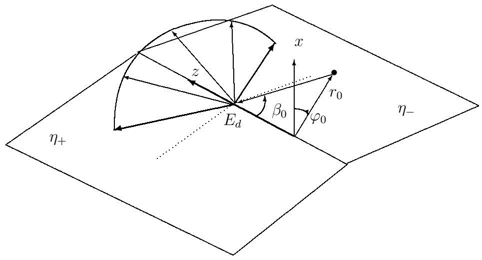

Figure 2: Edge wave and Keller's cone

components in the far-field asymptotics. We apply multidimensional versions of the saddle point technique. On this way, we are forced to consider two different cases. The first case implies that the point source is located outside a close neighbourhood of the wedge's faces. The second one corresponds to the source located near a wegde's face, however, outside some close vicinity of the edge. In the first case we obtain the asymptotic expressions of the far field that have clear physical meaning of the waves reflected from the faces. The other components correspond to the diffracted wave from the edge as well as the surface waves propagating from the edge (the latters are not considered herein, see [4]). These terms are analogous to those obtained in the electromagnetic version of the problem [1]. The asymptotic expressions are written in simple geometrical terms, depending on Malyuzhinets function, and can be easily interpreted in the framework of Geometrical Theory of Diffraction (GTD). The latter means that the geometrical optics objects like angle of conical diffraction, diffractional eikonals etc. can be easily identified in the far-field expressions. In the second case the results are similar. However, several new terms in the asymptotics arise. The source located near a wedge's face additionally generates the so-called primary surface wave that propagates to the edge and gives rise to the reflected and transmitted surface waves. We discuss the laws of reflection and transmission of these waves. On this way, an exaustive physical analysis of wave processes can be given. At the same time, the space wave from the edge is also generated as a result of interaction of the primary surface waves with the edge. Contrary to the result discussed in [3] that requires solution of an integral equation, we obtain new expressions for the diffraction or excitation coefficients which can be directly used for numerics. These expressions depend on the eponymous Malyuzhinets function and can be efficiently incorporated in various GTD type procedures of research and engineering practice.

3.1 The source is not close to the wedge's faces

It is easy to derive the primary spherical wave from the point source as well as those reflected from the faces. The surface waves are also excited and computed from the Weyl integral representation [4].

However, we turn to the edge wave induced by the interaction of the spherical wave from the source with the edge (Fig 2). Applying the formula for the leading term in the 3D steepest descent technique to the Weyl type integral (the details of the derivation can be found in [4])

$$u^e(M) = \frac{\mathrm{i}k}{8\pi^2}\int_{S(\beta_0)} \mathrm{d}\beta \int_{S(\varphi_0)} \mathrm{d}\alpha \int_{S_{\arg\sin(\beta)}(0)} \frac{\mathrm{d}s}{2\pi \mathrm{i}}$$
$$\sin\beta\, \mathrm{e}^{\mathrm{i}k[z\cos\beta+\sin\beta(r_0\cos(\alpha-\varphi_0)+r\cos s)]}\times$$
$$[f(s+\pi+\varphi;\alpha,\beta) - f(s-\pi+\varphi;\alpha,\beta)],$$

we arrive at[1] the nonuniform (w.r.t. angles of observation) expression for the edge wave

$$u^e(M) = \frac{\mathrm{e}^{\mathrm{i}\pi/4}}{4\pi}\frac{\mathrm{e}^{\mathrm{i}k\sqrt{z^2+(r+r_0)^2}}}{\sqrt{(z^2+(r+r_0)^2)}}\left\{\frac{\sqrt{z^2+(r+r_0)^2}}{2\pi k r r_0}\right\}^{1/2} \tag{4}$$
$$\mathscr{D}(\varphi,\varphi_0,\beta_0)\left(1+O\left(\frac{1}{k}\right)\right),$$

where $\mathscr{D}(\varphi,\varphi_0,\beta_0) = f(-\pi+\varphi;\varphi_0,\beta_0) - f(\pi+\varphi;\varphi_0,\beta_0)$ is the diffraction coefficient of the edge wave, β_0 is a solution of the equation $\cot\beta_0 = \frac{z}{r+r_0}$ from the segment $(0,\pi)$,

$$f(s;\alpha,\beta) = \frac{\mu\cos\mu\alpha}{\sin\mu s - \sin\mu\alpha}\,\frac{\Psi(s;\beta)}{\Psi(\alpha;\beta)},\ \ \mu = \frac{\pi}{2\Phi}$$

and

$$\Psi(s;\beta) =$$
$$\psi_\Phi(s-\Phi+\pi/2-\theta^-(\beta))\psi_\Phi(s-\Phi-\pi/2+\theta^-(\beta))\times$$
$$\psi_\Phi(s+\Phi+\pi/2-\theta^+(\beta))\psi_\Phi(s+\Phi-\pi/2+\theta^+(\beta)),$$

where ψ_Φ is the Malyuzhinets function, $\sin\theta^\pm(\beta) = \frac{\eta_\pm}{\sin\beta}$ and the branch of $\theta^\pm(\beta)$ is to be appropriately chosen [4].

3.2 The source is close to a wedge's face

Provided the point source is located near a wedge's face, as was mentioned above some additional components appear in the far field. The primary surface wave (see the corresponding expression in [4]) is excited on the face and propagates to the edge and gives rise to the edge wave and to the additional (w.r.t. those mentioned in the previous Section 3.1) reflected and transmitted surface wave (Fig. 3). The reflected surface wave is given by

$$u_r^{sw}(M) = r_{\theta^+}(\tau_+)\times$$
$$\frac{k\mathrm{e}^{\mathrm{i}3\pi/4}}{2\sqrt{2\pi}}\frac{\mathrm{e}^{\mathrm{i}k[r_0\sin(\varphi_0-\Phi)\eta_+ + r\sin(\varphi-\Phi)\eta_+]}}{\sqrt{(1-\eta_+^2)k\rho}}\left\{\frac{1-\eta_+^2}{1+z^2/\rho^2}\right\}^{3/4} \tag{5}$$
$$\mathrm{e}^{\mathrm{i}k\sqrt{(1-\eta_+^2)(z^2+\rho^2)}}\left(1+O\left(\frac{1}{k}\right)\right),$$

where we use the notations $r_{\theta^+}(\tau) = R_{\theta^+}(\beta)\mathrm{res}_{\Phi+\theta^+(\beta)}\, f(\Phi-\pi-\theta^+(\beta);\alpha,\beta)|_{\tau=\cos\beta}$ and $\rho = r_0\cos[\Phi-\varphi_0] + r\cos[\Phi-\varphi]$, $R_{\theta^\pm}(\beta) = \pm 2\tan\theta^\pm(\beta)$. The GO law of reflection of the surface wave at the edge implies that the angle of incidence is equal to that of reflection.

The leading term of the transmitted surface wave takes the form

$$u_t^{sw}(M) = r_{\theta^-}(\tau_-)\frac{k\mathrm{e}^{\mathrm{i}3\pi/4}}{2\sqrt{2\pi}}\frac{\mathrm{e}^{\mathrm{i}k[r_0\sin(\varphi_0-\Phi)\eta_- + r\sin(\varphi-\Phi)\eta_-]}}{\sqrt{k\rho}}\times$$
$$\left\{\frac{\gamma^+(1-\eta_+^2)\tau_-}{[\sqrt{1-\eta_+^2-\tau_-^2}]^3} + \frac{\gamma^-(1-\eta_-^2)\tau_-}{[\sqrt{1-\eta_-^2-\tau_-^2}]^3}\right\}^{-1/2}\times \tag{6}$$
$$\mathrm{e}^{\mathrm{i}k[z\tau_- + \rho\gamma^+\sqrt{1-\eta_+^2-\tau_-^2} + \rho\gamma^-\sqrt{1-\eta_-^2-\tau_-^2}]}\left(1+O\left(\frac{1}{k}\right)\right),$$

where $\gamma^+ = r_0\cos(\Phi-\varphi_0)/\rho$, $\gamma^- = r\cos(\varphi+\Phi)/\rho$. We attributed some geometrical meaning to the equation for the stationary point introducing the angles of incidence κ_+ and of refraction κ_- of the surface wave (Fig. 3)

$$\tan\kappa_\pm = \frac{\tau_-}{\sqrt{1-\eta_\pm^2-\tau_-^2}}$$

and writing the equation for the stationary point of the corresponding Weyl type integral as

$$z = r\cos(\varphi+\Phi)\tan\kappa_- + r_0\cos(\Phi-\varphi_0)\tan\kappa_+\,.$$

The Snel's type law of refraction of the primary surface wave across the edge of two impedance halfplanes reads

$$\sqrt{1-\eta_+^2}\sin\kappa_+ = \sqrt{1-\eta_-^2}\sin\kappa_-\,.$$

In the assumption $|\eta_-| < |\eta_+|$ the phenomenon of the total internal reflection of the primary surface wave arriving at the edge may occur.[2] The corresponding critical angle of the total internal reflection is given by ($\kappa_- \to \pi/2$)

$$\kappa_+^* = \arcsin\left(\sqrt{\frac{1-\eta_-^2}{1-\eta_+^2}}\right).$$

For the critical angle $\kappa_+ = \kappa_+^*$ the transmitted surface wave propagates along the edge ($\kappa_- = \pi/2$) and also exponentially vanishes as $kr \to \infty$. Remark that in the formulas (5),(6) the coefficients $r_{\theta^\pm}$ are expressed in terms of the Malyuzhinets function.

We turn to the edge wave excited by the primary surface wave (Fig. 4). Introduce the angles of diffraction and of incidence of the surface wave which are defined by

$$\tan\kappa_d = \frac{\tau_e}{\sqrt{1-\tau_e^2}},\ \ \tan\kappa_0 = \frac{\tau_e}{\sqrt{1-\eta_+^2-\tau_e^2}}.$$

[1]The formula for the leading term of the asymptotics in the electromagnetic case is given in [1] pp. 89-92.

[2]Recall that $\eta_\pm$ are assumed to be purely imaginary.

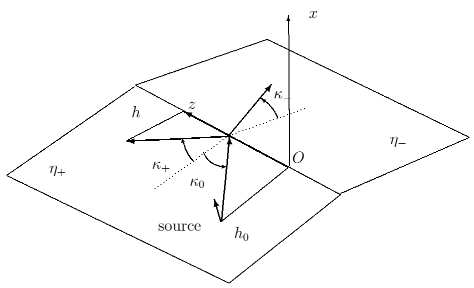

Figure 3: Reflected and transmitted surface waves by the edge

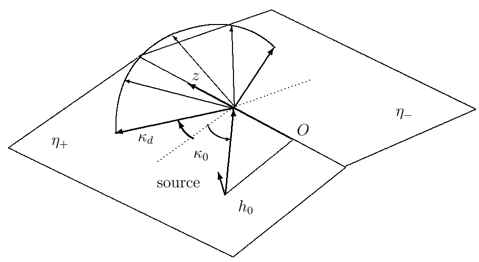

Figure 4: The edge wave excited by the primary surface wave

The leading term for the edge wave excited by the primary surface wave reads ($\tau_e = \cos\beta_e$)

$$u_e^{sw}(M) = \frac{d(\tau_e,0;\varphi)}{4\pi}\left\{\frac{r^2}{\sin^2\beta_e} + rr_0\cos[\Phi-\varphi_0]\frac{(1-\eta_+^2)\sin\beta_e}{\sqrt{\sin^2\beta_e-\tau_e^2}}\right\}^{-1/2}$$
$$\mathrm{e}^{-\mathrm{i}kr_0\sin(\Phi-\varphi_0)\eta_+}\,\mathrm{e}^{\mathrm{i}k[z\cos\beta_e+\sin\beta_e r+r_0\cos(\Phi-\varphi_0)\sqrt{\sin^2\beta_e-\eta_+^2}]}\times$$
$$\left(1+O\left(\tfrac{1}{k}\right)\right). \tag{7}$$

Remark that the formula (7) (compare with (4)) can also be written in terms of the angles κ_0 and κ_d, $d(\tau,s;\varphi) = \mathrm{res}_{\alpha=\Phi+\theta^+(\beta)}[f(s+\pi+\varphi;\alpha,\beta) - f(s-\pi+\varphi;\alpha,\beta)]|_{\tau=\cos\beta}$.

The Geometrical Theory of Diffraction (of Keller with abbriviation GTD) traditionally operates with various types of rays: incident, reflected, diffracted or others. In our case, the diffracted rays (analogous to those in Fig. 2) compose the Keller cone the opening of which is $\pi/2-\kappa_d$, where κ_d is determined from the transcendental equation

$$\frac{\sin\kappa_d}{\sqrt{\cos^2\kappa_d-\eta_+^2}} = \tan\kappa_0$$

if the angle of incidence κ_0 is given. In GTD the latter equation can naturally be called the law of conical diffraction of the surface wave at the edge of an impedance wedge. It obviously has an asymptotic nature and can be exploited together with high-frequency localization principle. It is worth noticing the existence of the critical angle κ_0^* for edge wave which corresponds to $\kappa_d = \pi/2$,

$$\kappa_0^* = \arctan\left(\frac{1}{|\eta_+|}\right).$$

For this angle the edge wave collapses to be concentrated near the edge.

4 Acknowledgemnts

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