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# Towards a hierarchical representation of the embedded element patterns of phased arrays

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The imaging and calibration techniques for the next radio telescopes composed of phased arrays, such as the Square Kilometer Array, will have to account for the mutual coupling between the antennas in order to meet their dynamic range requirements. In this context, the embedded element patterns (EEP), which can be viewed as direction-dependent antenna gains, need to be evaluated at a number of directions $N_{dir}$ proportional to the number of antennas $N_a$ since their spatial bandwidth is given by the electrical size of the array. Hence, the computational costs of basic operations with the EEPs, e.g. beamforming or cross-correlations, are high when the array contains many antennas since they scale at least as $N_a^2$. We propose here a hierarchical representation of the EEPs based on a fast direct solver which allows to perform arithmetic operations with a reduced asymptotic complexity.
Let us first recall the Method-of-Moments system of equations, $\mathbf{Z}\mathbf{i}_n = \mathbf{v}_n$ where the induced currents $\mathbf{i}_n$ are discretized with $M$ basis functions per antenna, $\mathbf{Z}$ is the $MN_a \times MN_a$ impedance matrix and $\mathbf{v}_n$ is the $MN_a \times 1$ excitation vector when the antenna $n$ is fed by a voltage source. From there, the EEP of polarization $p$ of antenna $n$ evaluated at $N_{dir}$ directions is computed with

$$\mathbf{f}_{p,n} = \mathbf{F}_p \mathbf{Z}^{-1} \mathbf{v}_n \tag{1}$$

where $\mathbf{f}_{p,n}$ is a $N_{dir} \times 1$ vector and $\mathbf{F}_p$ is a $N_{dir} \times N_a M$ Fourier-type matrix transforming currents into far fields. The costly operation in (1) is the product of $\mathbf{Z}^{-1}$ with the excitation vector $\mathbf{v}_n$. This product can be expedited in $N_a \log^2 N_a$ with low overheads using a fast direct solver [1] which exploits the reduced degrees of freedom of on-plane fields scattered from elongated structures [1], [2]. The fast method first decomposes the array geometry into a quadtree of depth $L \propto \log_4 N_a$ whose leaves contain one antenna. Then, the impedance matrix $\mathbf{Z}$ is hierarchically factorized into the sum of a diagonal matrix $\mathbf{Z}_{d,l}$ containing the near-field interations between the child boxes of level $l$ and a low-rank product approximating the rank-defficient far interactions between boxes of level $l$ using an interpolative decomposition [1]. This leads to the following recursive relation at each level $l$,

$$\mathbf{Z}_l \approx \mathbf{Z}_{d,l} + \mathbf{Q}_l^T \mathbf{Z}_{l-1} \mathbf{Q}_l \tag{2}$$

where $\mathbf{Z}_{d,l}$ and $\mathbf{Q}_l$ are block-diagonal matrices and $\mathbf{Z}_{l-1}$ is a skeleton matrix containing elements of the off-diagonal blocks of $\mathbf{Z}_l$. The recursion terminates with a matrix $\mathbf{Z}_0$ of size $N_0 \propto \sqrt{N_a M} \log_2 N_a M$ [2] and an approximate $\mathbf{Z}_L$ of $\mathbf{Z}$. The bottom line of the fast direct solver is that a similar factorization for $\mathbf{Z}^{-1}$ can be obtained directly from that of $\mathbf{Z}$. One can infer from (2) that the product of $\mathbf{Z}_L$ or $\mathbf{Z}_L^{-1}$ with any vector is performed in $N_a M \log^2 N_a M$ operations.
The method is illustrated for a regular array of simplified SKALA4 antennas [3] separated by 2m and meshed with 50 basis functions at 80 MHz. The number of antennas is progressively increased from 4 to 256. The relative error of the approximate matrices is computed with $\varepsilon_z = ||\mathbf{Z}_L - \mathbf{Z}||_F / ||\mathbf{Z}||_F$ with $||.||_F$ the Frobenius norm. The number $L$ of tree levels, the number $N_a$ of antennas, the error $\varepsilon_z$ of the approximate impedance matrix $\mathbf{Z}_L$, the error $\varepsilon_{z^{-1}}$ of the approximate inverse matrix $\mathbf{Z}_L^{-1}$ and the size $N_0$ of the compressed matrices $\mathbf{Z}_0$ and $\mathbf{Z}_0^{-1}$ are, respectively, given by $L = \{1,2,3,4\}$, $N_a = \{4,16,64,256\}$, $\varepsilon_z = \{1.2,1.5,2.2,3\}10^{-4}$, $\varepsilon_{z^{-1}} = \{2.9,2,3,4.2\}10^{-3}$ and $N_0 = \{40,100,210,426\}$. As expected, this shows that $N_0$ is almost proportional to $\sqrt{N_a}$ and that the error $\varepsilon_{z^{-1}}$ is one order of magnitude higher than $\varepsilon_z$. Those preliminary results allow us to foreseen important speed-ups in imaging and calibration algorithms thanks to this hierarchical representation of the EEPs.

## References

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