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High-frequency Lengthwise Diffraction by the Line Separating Soft and Hard Parts of the Surface

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Abstract

The paper examines the model problem of high-frequency diffraction by a convex surface consisting of two parts. One is soft, the other is hard. The incident wave falls at a small angle to the line which separates soft and hard parts of the surface. The change in the boundary condition provokes the field in the Fock zone to have a quick transverse variation. This causes a special boundary-layer to be formed. The boundary value problem for the three dimensional parabolic equation is reduced to the Riemann problem solved by the factorization in the form of infinite products containing the zeros of the Airy function and zeros of its derivative. the results of this factorization appear under the sign of double Fourier integral in the representation of the field. Both numerical and asymptotic analysis of this representation is carried out and illustrates the effects of high-frequency diffraction caused by the line of the boundary condition discontinuity.

1 Introduction

In high frequency diffraction by a smooth convex obstacle the wave field on the shadowed side of the body is formed by creeping waves [1, 2]. Creeping waves are launched by the incident wave at light-shadow boundary on the surface, where a specific boundary layer is formed. If the surface of the body has discontinuities (edges, wedges, jumps of curvature or jumps in the boundary impedance) creeping waves may diffract on them. Analysis of the influence of the discontinuities of the surface have been studied in [3] and many later papers. Effects of edge diffraction are described by means of hybrid diffraction coefficients [4] in the frame of the physical theory of diffraction. More rigorous mathematical considerations can be found in [5]. Effects of curvature discontinuities at a junction with flat surface are studied in [6], and with a concave boundary in [7]. Many other papers consider the effects of curvature and higher order discontinuities of the surface. The effects of the impedance discontinuity have been studied in [8]. More specific effects caused by the impedance taking particular values are studied in [9, 10]. All these results are essentially two-dimensional and in scaled boundary layer coordinates the incidence at a line of such discontinuity becomes effectively orthogonal. However, if a creeping wave runs almost parallel to such edges or lines of curvature or impedance discontinuities, the effects of diffraction become essentially three-dimensional and up to our knowledge are not discussed in literature.

This paper considers a model problem of diffraction by a convex surface consisting of two parts. One is soft, the other is hard. This is the limit case of discontinuous impedance. We assume that the curve γ, which separates the two parts of the surface, is almost a straight line (geodesics) and it is almost parallel to the direction of creeping wave propagation. We consider the vicinity of the light-shadow boundary on the surface, so-called Fock zone and develop there the parabolic equation method.

2 The problem of diffraction

We examine diffraction of a high-frequency scalar wave on a smooth convex surface Γ. The ray field expansion of the incident wave u^i induces a semi-geodesic surface coordinate system (s,a,n). The normal n is directed to the convex side. The projection of the observation point on the surface is defined by the coordinate a which determines the particular geodesic line and the distance s measured from some reference line $s=0$. The light-shadow boundary is defined by the equation $s=s_0(a)$ and all the geodesics are directed along the incident ray which is tangent to Γ at $s=s_0(a)$. Let the surface consists of two parts Γ_1 and Γ_2 separated by line γ from each other. Let the part Γ_1 be absolutely soft and the part Γ_2 be absolutely hard. The geometry of the problem is shown on Fig. 1. Let the line γ be given by the equation

$$a=a_0(s).$$

We consider such case when the curve γ is close to a geodesic curve on the surface and the angle between γ and the system of geodesic lines $a=const$ is small. More precisely, $\rho a_0''(0)=o(1)$ and $\psi=ma_0'(0)<\psi_0=O(1)$. Here $m=(k\rho/2)^{1/3}$, k is the wavenumber and ρ is the radius of curvature of the geodesics $a=0$ at its crossing with the light-shadow boundary, i.e. at $s=0$.

The stationary wave field u is subject to Helmholtz equation

$$\triangle u+k^2u=0$$

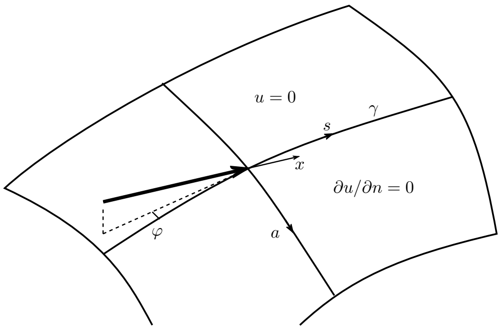

Figure 1. Geometry of the problem.

in the exterior of the surface Γ and the boundary conditions

$$u|_{\Gamma_1} = 0, \qquad \left.\frac{\partial u}{\partial n}\right|_{\Gamma_2} = 0.$$

To conclude the problem formulation we add radiation conditions at infinity and Meixner conditions on the curve γ.

We consider the field u in penumbra where additionally to the usual [1, 2] stretched coordinates

$$\sigma = m\frac{s}{\rho}, \qquad \nu = 2m^2\frac{n}{\rho}$$

we stretch also the transverse coordinate

$$\alpha = 2m^2\frac{a}{\rho} - 2\psi\sigma.$$

For the attenuation function $U(\nu,\sigma,\alpha)$ this results in the boundary value problem for the parabolic equation

$$i\frac{\partial U}{\partial \sigma} + \frac{\partial^2 U}{\partial \nu^2} + \frac{\partial^2 U}{\partial \alpha^2} - 2i\psi\frac{\partial U}{\partial \alpha} + \nu U = 0 \qquad (1)$$

with the boundary conditions

$$U(0,\sigma,\alpha) = 0, \qquad \alpha < 0, \qquad (2)$$

$$\frac{\partial U(0,\sigma,\alpha)}{\partial \nu} = 0, \qquad \alpha > 0. \qquad (3)$$

3 The Riemann problem

It is convenient to extract from the solution U the attenuation function corresponding to the hard surface [1, 2]

$$U_1 = \frac{1}{\sqrt{\pi}}\int e^{i\lambda\sigma}\left\{v(\lambda-\nu) - \frac{v'(\lambda)}{w_1'(\lambda)}w_1(\lambda-\nu)\right\}d\lambda.$$

Here v and w_1 are the Airy functions in Fock notations, i.e. $v() = \sqrt{\pi}\mathrm{Ai}()$, $w_1(z) = 2e^{\pi i/6}v(e^{2\pi i/3}z)$. The correction $U_2 = U - U_1$ is searched for in the form of double Fourier transform

$$U_2 = \iint e^{i\lambda\sigma - i\eta\alpha}A(\lambda,\eta)w_1\left(\lambda+\eta^2-2\psi\eta-\nu\right)d\lambda\, d\eta. \qquad (4)$$

By changing integration variables in (4) to $\xi = \lambda - \psi^2$ and $\mu = \eta - \psi$ and introducing the new unknown

$$Q^{(-)}(\mu;\xi) = A(\lambda,\eta)w_1'(\xi+\mu^2), \qquad (5)$$

which is an analytic function in the lower half-plane of μ, the boundary value problem (1)–(3) is reduced to the Riemann problem of factorization:

$$Q^{(-)}(\mu;\xi)G(\mu;\xi) = Q^{(+)}(\mu;\xi) - \frac{H(\mu;\xi)}{2i\pi^{3/2}}, \qquad (6)$$

where

$$G = \frac{w_1(\xi+\mu^2)}{w_1'(\xi+\mu^2)}, \quad H = \frac{1}{w_1'(\xi+\mu^2)}\frac{1}{\mu+\psi-i0}.$$

The factorization in (6) should be performed with respect to variable μ, while ξ plays the role of a parameter.

To solve (6) it is necessary to represent the symbol G as the product of functions analytic in the upper/lower half planes of μ. Using Weierstrass theorem for entire functions we get

$$w_1(z) = w_1(0)\exp\left(\frac{w_1'(0)}{w_1(0)}z\right)\prod_n\left(1-\frac{z}{t_n}\right)\exp\left(\frac{z}{t_n}\right), \quad (7)$$

$$w_1'(z) = w_1'(0)\prod_n\left(1-\frac{z}{t_n'}\right)\exp\left(\frac{z}{t_n'}\right), \qquad (8)$$

where t_n and t_n' are the zeros of the Airy function w_1 and its derivative w_1' correspondingly. These zeros have the asymptotic approximations [11]

$$t_n = e^{i\pi/3}\left(\frac{3\pi}{2}\left(n-\frac{1}{4}\right)\right)^{2/3}, \quad t_n' = e^{i\pi/3}\left(\frac{3\pi}{2}\left(n-\frac{3}{4}\right)\right)^{2/3}. \qquad (9)$$

The asymptotic approximations (9) show that after combining representations (7) and (8) the exponential factors can be extracted out from the infinite product and summed up separately. In view of the identity

$$\frac{w_1'(0)}{w_1(0)} + \sum_n\left(\frac{1}{t_n} - \frac{1}{t_n'}\right) = 0$$

this yields the following representation for the symbol

$$G(\mu;\xi) = \frac{w_1(\xi+\mu^2)}{w_1'(\xi+\mu^2)} = \frac{w_1(0)}{w_1'(0)}\prod_n\frac{t_n'}{t_n}\frac{t_n-\xi-\mu^2}{t_n'-\xi-\mu^2}.$$

It allows the factorization in the form

$$G(\mu;\xi) = \frac{w_1(\xi)}{w_1'(\xi)}G^{(+)}(\mu;\xi)G^{(-)}(\mu;\xi), \qquad (10)$$

where

$$G^{(\pm)}(\mu;\xi) = \prod_n\frac{1\pm\frac{\mu}{\sqrt{t_n-\xi}}}{1\pm\frac{\mu}{\sqrt{t_n'-\xi}}}. \qquad (11)$$

Function $Q^{(-)}$ in (6) can be expressed via $G^{(\pm)}$ explicitly:

$$Q^{(-)} = \frac{i}{2\pi^{3/2}}\frac{w_1(\xi)}{w_1'(\xi)w_1(\xi+\psi^2)}\frac{G^{(+)}(\psi;\xi)}{G^{(-)}(\mu;\xi)}\frac{1}{\mu+\psi-i0}.$$

Substituting it in (5) and (4) gives the leading order approximation for the field, which we discuss in the next section.

4 The leading order approximation

The high-frequency asymptotic approximation for the total field at the leading order can be written as

$$u = \frac{e^{iks+i\psi^2\sigma}}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{i\sigma\xi} \Bigg\{ v(\xi+\psi^2-\nu) - \frac{v'(\xi+\psi^2)}{w_1'(\xi+\psi^2)} w_1(\xi+\psi^2-\nu) + \frac{i}{2\pi}\frac{1}{w_1'(\xi+\psi^2)} \int_{-\infty}^{+\infty} e^{-i\alpha\mu-i\psi\alpha} \frac{G^{(+)}(\mu,\xi)}{G^{(-)}(\psi,\xi)} \times \times \frac{w_1(\xi+\mu^2-\nu)}{w_1(\xi+\mu^2)} \frac{d\mu}{\mu+\psi-i0} \Bigg\} d\xi. \quad (12)$$

It can be easily checked that the approximation (12) satisfies the boundary conditions (2) and (3). Indeed, by setting $\nu = 0$ we get the integral with respect to μ of a function having only one pole $\mu = -\psi + i0$ in the upper half-plane. Thus for $\alpha < 0$ it reduces to $2\pi i$ and in view of $v'w_1 - vw_1' = 1$ compensates the first two terms, resulting in $u = 0$. Differentiating in (12) with respect to ν yields the expression

$$G^{(+)}(\mu;\xi)\frac{w_1'(\xi+\mu^2)}{w_1(\xi+\mu^2)}$$

which in view of representation (10) is analytic in the lower half-plane of μ yielding the integral to be zero for $\alpha > 0$. Which shows that the boundary condition (3) is also satisfied.

Numerical results for the field and its derivative with respect to ν on the surface are shown on Fig. 2. These graphics show that when $|\alpha|$ increases the amplitudes of the field and its normal derivative tend to the values corresponding to the soft and to the hard surface (These values are marked with bullets on the left and right axes). For small $|\alpha|$ the computations are not correct, which is a purely numerical problem caused by very slow convergence of the products in (11). Analytical analysis of the Fourier integral shows that the field behaves as $\sqrt{\alpha}$ and $\partial u/\partial\nu$ is inverse proportional to $\sqrt{-\alpha}$.

5 The field in the deep shadow

The asymptotic representation of the integral (12) for $\sigma \to +\infty$ and bounded α manifests contributions of the singularities, which are:

- poles at $\xi = t_n' - \psi^2$ if $\psi < 0$;
- poles at $\xi = t_n - \psi^2$ if $\psi > 0$;
- branch-points at $\xi = t_n'$ and $\xi = t_n$.

The contributions of the poles are the usual creeping waves on the hard and on the soft parts of the surface. In the case of $\psi > 0$, every soft surface creeping wave when it reaches

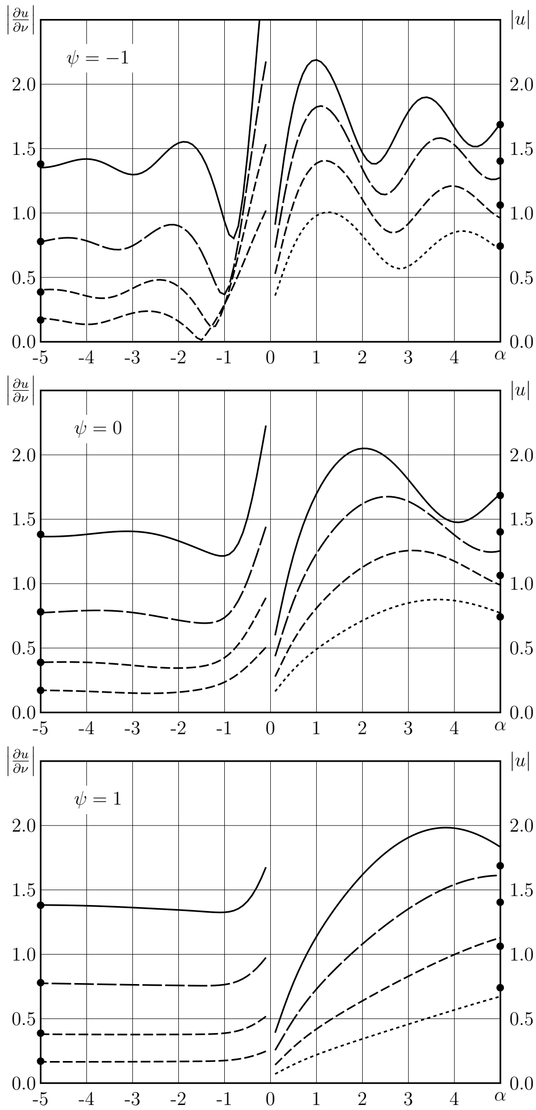

Figure 2. The field and its normal derivative on the surface. Solid, long-dashed, medium-dashed and short-dashed curves present fields at $\sigma = -0.5$, 0, 0.5 and 1 correspondingly.

the line γ launches the infinite series of "reflected" creeping waves and the infinite series of "transmitted" hard surface creeping waves. The directions of their propagation are defined by the requirement of equal phases on γ. Analogously, in the case of $\psi < 0$ every hard surface creeping wave launches the series of "reflected" hard surface creeping waves and the series of "transmitted" soft surface creeping waves.

The branch-points contributions have the form

$$\frac{H_n}{\sigma^{3/2}} \exp\left(iks + it_n'\sigma + i\psi^2\sigma - i\psi\alpha\right),$$

$$\frac{S_n}{\sigma^{3/2}} \exp\left(iks + it_n\sigma + i\psi^2\sigma - i\psi\alpha\right),$$

where H_n and S_n are some complicated expressions depending on α. In addition to the usual creeping waves decay described by the exponentials $\exp(-\Im(t'_n)\sigma)$ or $\exp(-\Im(t_n)\sigma)$ these waves have power attenuation described by the factor $\sigma^{-3/2}$, which resembles head waves. In the case of $\psi = 0$ creeping waves disappear, but head waves get the factor $\sigma^{-1/2}$ instead of $\sigma^{-3/2}$.

6 Conclusion

We have constructed the leading order term asymptotic approximation (12) for the scalar field on the surface consisting of soft and hard parts. This expression is valid in the case of small angle between the direction of incidence and the line γ separating soft and hard parts of the surface. Nonuniform asymptotic simplifications of this cumbersome expression shows specific effects of diffraction which take place in a vicinity of the point where γ crosses the light-shadow boundary.

Similar effects should have place in the case of electromagnetic field diffraction, though the expressions for the fields are expected to be more cumbersome. Another generalization of the approach is possible for the case of the impedance boundary conditions with two different impedances on the parts of the surface.

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